13.1 Vectors: Displacement Vectors

Review of vectors:
Some physical quantities can be completely defined by magnitudes (speed, mass, length, time, etc.) These are called scalars. Other quantities need a magnitude and direction to be completely defined (displacement, velocity, force, etc.) These are called vectors.

The displacement vector \( \vec{d} \) from one point to another is an arrow with its tail at the first point and its tip at the second point. The magnitude of the displacement vector \( |\vec{d}| \), denoted by \( \vec{v} \), is the distance between the points and represented by the length of its arrow. The direction of the displacement vector is the direction of its arrow, and it is the angle measured from the positive \( x \)-axis.

Properties of Vectors
1. Two vectors are equal \( \vec{v} = \vec{u} \), if they have the same magnitude and direction.
2. The negative of the vector \( \vec{v} \) is a vector with same magnitude but opposite direction.
3. The product of a vector \( \vec{v} \) with a constant \( k \) is a vector with magnitude \( k \) times the magnitude of \( \vec{v} \) in, the direction of \( \vec{v} \).
4. The zero-vector is a vector with no length and no specified direction.
5. We add two vectors geometrically by using the parallelogram law. Place the tail of the second vector at the tip of the first vector, and connect the tail of the first vector at the tip of the second vector with another vector. This new vector is called the resultant vector.
6. To subtract two vectors geometrically place the two tails of the vectors together and connect the two tips with another vector. This vector is in the direction tip minus tail.

Vectors in Coordinate Systems on the Plane (2-space)
1. A position vector has its initial point at the origin of the coordinate system and the final point at some other point on the coordinate plane.
2. A position vector can be expressed by listing its components (coordinates of the tip of a position vector). \( \vec{v} = <v_1, v_2> \)
   eg. The vector \( <2, 4> \) can be interpreted as a position vector that points to the point \( (2, 4) \) on the plane.
3. Any vector can be made a position vector. If the vector \( \vec{w} = < w_1, w_2 > \) from the point 
\( P_1 = (x_1, y_1) \) to the point \( P_2 = (x_2, y_2) \) is not at the origin, then \( \vec{w} = < x_2 - x_1, y_2 - y_1 > \) where \( P_1 \) is the initial point, and \( P_2 \) is the final point (tip minus tail). This is the same as subtracting two position vectors one at the tip the other at the tail. \( \vec{w} = \vec{u} - \vec{v} \) where \( \vec{u} \) is the tip vector and \( \vec{v} \) is the tail vector.

e.g. The vector from point \((2, 3)\) to point \((5, -2)\), can be represented as \(< 3, -5 >.\)

If you subtract the position vector \( \vec{u} = < 2, 3 > \) from vector \( \vec{v} = < 5, -2 >, \) the vector \( \vec{u} - \vec{v} = < 3, -5 >. \)

4. The zero-vector is expressed \( \vec{0} = < 0,0 >. \)

5. To add or subtract two vectors algebraically, add the respective components. If \( \vec{u} = < u_1, u_2 >, \)
and \( \vec{v} = < v_1, v_2 >, \) then \( \vec{u} \pm \vec{v} = < u_1 \pm v_1, u_2 \pm v_2 > \)
e.g. Draw the vectors \( \vec{u} = < 3, 1 >, \vec{v} = < 2, 4 > \) and \( \vec{u} \pm \vec{v}. \)

6. If a vector is multiplied by a constant \( k \) (scalar multiplication) then each component is multiplied by \( k. k \vec{v} = < kv_1, kv_2 > \)

End Review of Vectors

Vectors in coordinate systems in space (3-space)

1) A position vector has its initial point at the origin of the coordinate system.
2) A position vector can be expressed by listing its components (coordinates of the tip of a position vector).
\( \vec{v} = < v_1, v_2, v_3 >. \)
3) Any vector can be made a position vector. If the vector \( \vec{w} = < w_1, w_2, w_3 > \) is not at the origin, but from the point \( P_1 = (x_1, y_1, z_1) \) to the point \( P_2 = (x_2, y_2, z_2) \), then
\( \vec{w} = < x_2 - x_1, y_2 - y_1, z_2 - z_1 > \) where \( P_1 \) is the initial point, and \( P_2 \) is the final point. (tip minus tail). This is the same as subtracting two position vectors one at the tip the other at the tail. \( \vec{w} = \vec{u} - \vec{v} \) where \( \vec{u} \) is the tip vector and \( \vec{v} \) is the tail vector.
4) The zero-vector is expressed \( \vec{0} = < 0,0,0 >. \)
5) To add or subtract two vectors add the respective components. If \( \vec{u} = < u_1, u_2, u_3 >, \)
and \( \vec{v} = < v_1, v_2, v_3 >, \) then \( \vec{u} \pm \vec{v} = < u_1 \pm v_1, u_2 \pm u_2, u_3 \pm v_3 >. \)
6) If a vector is multiplied by a constant \( k \) (scalar multiplication) then each component is multiplied by \( k. k \vec{v} = < kv_1, kv_2, kv_3 >. \)

Norm or magnitude (length) of a vector

The norm of a vector from the point \( P_1 = (x_1, y_1) \) to the point \( P_2 = (x_2, y_2), \) in 2-space, is given by the distance formula as
\[ |\vec{v}| = \sqrt{v_1^2 + v_2^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
The norm of a vector from the point \( P_1 = (x_1, y_1, z_1) \) to the point \( P_2 = (x_2, y_2, z_2) \), in 3-space, is given by the distance formula as:

\[
|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\]

**Unit vectors on the plane**

A unit vector is a vector with magnitude of one unit. \( \hat{v} = \frac{\vec{v}}{|\vec{v}|} \). A unit vector gives direction.

Vectors of importance are the unit vectors in the direction of the coordinate axis
\( \hat{i} = <1, 0> \); \( \hat{j} = <0, 1> \).

Any vector can be expressed in terms of the unit vectors \( \hat{i}, \hat{j} \).

**eg1** Express \( <2, 5> \) as a vector in terms of \( \hat{i}, \hat{j} \).
\( <2, 5> = 2\hat{i} + 5\hat{j} \)
Check: \( 2\hat{i} + 5\hat{j} = 2<1, 0> + 5<0, 1> = <2, 5> \).

**eg2** Express \( <2, 5> \) as a unit vector.
\[
|<2, 5>| = \sqrt{29}, \text{ so } <\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}> \text{ is a unit vector.}
\]

**eg3** Find a vector 3 units long in the direction of \( \vec{A} = <2, -1> \).
\[
3\vec{A} = \frac{3}{\sqrt{5}} <2, -1> = \frac{6}{\sqrt{5}} <6, -3>.
\]

Any vector can be expressed as a product of its magnitude and its direction.
\[
\vec{r} = |\vec{r}| \hat{r} = |\vec{r}| \hat{r}
\]

**eg4** Express \( \vec{r} = <2, 5> \) as a product of its magnitude and its direction.
\[
\vec{r} = <2, 5> = \sqrt{29} <\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}> = |\vec{r}| \hat{r}, \text{ where } \sqrt{29} \text{ is the magnitude of the vector } <2, 5>, \text{ and }
\]
\[
<\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}> \text{ is the unit vector or direction of the vector } <2, 5>.
\]

**Unit vectors in space:**

Vector with magnitude of one unit \( \hat{v} = \frac{\vec{v}}{|\vec{v}|} \). The vectors in the direction of the coordinate axis are
\( \hat{i} = <1, 0, 0> \); \( \hat{j} = <0, 1, 0> \); \( \hat{k} = <0, 0, 1> \).

**eg5** Give the direction of the vector \( \vec{v} = <4, -4, 2> \)
Since \( \vec{v} = <4, -4, 2> = 2 <2, -2, 1> \), \( \hat{v} = \frac{2 <2, -2, 1>}{2\sqrt{3}} = <\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}> \) is the direction of the vector \( \vec{v} \).

Any vector in space can be expressed in terms of the unit vectors \( \hat{i}, \hat{j}, \hat{k} \).
\[ \mathbf{e6} \mathbf{< 2,5,-1 >} = 2 \mathbf{j} + 5 \mathbf{j} - \mathbf{k} = 2 \mathbf{< 1,0,0 >} + 5 \mathbf{< 0,1,0 >} - \mathbf{< 0,0,1 >} = \mathbf{< 2,0,0 >} + \mathbf{< 0,5,0 >} - \mathbf{< 0,0,1 >} = \mathbf{< 2,5,-1 >} \]

Any vector in space can be expressed as a product of its magnitude and its direction.

\[ \mathbf{r} = |\mathbf{r}| \mathbf{\hat{r}} = |\mathbf{r}| \mathbf{\hat{r}} \]

**Vectors in Polar Coordinates**

A vector \( \mathbf{r} = < x, y > \) can be expressed in polar coordinates as \( \mathbf{r} = < r \cos \theta, r \sin \theta > \) where \( r = |\mathbf{r}| \) and \( \theta \) is the angle with the positive \( x \)-axis.

The unit vector \( \mathbf{\hat{r}} \) becomes \( \mathbf{\hat{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{< r \cos \theta, r \sin \theta >}{r} = < \cos \theta, \sin \theta > \).

**eg7** Find the position vector \( \mathbf{v} \) with magnitude of 2 and angle of inclination of \( -\frac{4\pi}{3} \).

\[ \mathbf{v} = < 2 \cos \left( \frac{-4\pi}{3} \right), 2 \sin \left( \frac{-4\pi}{3} \right) > = < -1, \sqrt{3} >. \]

**eg8** Find the unit vector that makes an angle of \( \frac{3\pi}{4} \) with the horizontal axes.

\[ \mathbf{\hat{r}} = < \cos \theta, \sin \theta > = < \cos \frac{3\pi}{4}, \sin \frac{3\pi}{4} > = < -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} > = -\frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j}. \]

**eg9** Give the angle of inclination of the vector \( < -\sqrt{3}, -1 > \), and represent the vector in polar coordinates.

Since this vector is a position vector in the third quadrant, the angle will be \( \tan \theta = \frac{1}{\sqrt{3}} \), \( \theta = \frac{7\pi}{6} \) in QIII. Since the magnitude of the vector is 2, so the desired vector is \( < 2 \cos \left( \frac{7\pi}{6} \right), 2 \sin \left( \frac{7\pi}{6} \right) >. \)

**Homework 13.1**

1. The figure below shows force vectors \( \mathbf{F}, \mathbf{G}, \) and \( \mathbf{H} \) and their \( x \)- and \( y \)- components. Calculate their resultant \( \mathbf{F} + \mathbf{G} + \mathbf{H} \) and add it to the drawing.

![Diagram of vectors](image)

\( \mathbf{F} = \langle -3, 6 \rangle \)

\( \mathbf{G} = \langle 4, 4 \rangle \)

\( \mathbf{H} = \langle 5, 1 \rangle \)

2. Give the two vectors of length 10 that are perpendicular to \( \langle -3, 4 \rangle \). Then draw the three vectors.

3. Find an angle of inclination of \( \langle -5, -3 \rangle \). Give an exact answer.
4. Find numbers $a$ and $b$ such that $a \langle 3, -1 \rangle + b \langle 1, 2 \rangle = \langle 1, -12 \rangle$.

5. What vector of length 7 has the same direction as $\overrightarrow{PQ}$ where $P = (-5, -3)$ and $Q = (4, -8)$?

6. Three ropes are supporting a 10 lb. weight. Two of the ropes exert forces $\vec{u} = \langle 2, 3, 4 \rangle$ lb and $\vec{v} = \langle -1, -2, 3 \rangle$ lb in the $xyz$ space with the positive $z$-axis pointing up. What is the tension in the third rope?

7. Consider the vector $\vec{v} = -\hat{i} + 2\hat{j} + 7\hat{k}$. Find a vector that
   a. Is parallel but not equal to $\vec{v}$.
   b. Points in the opposite direction of $\vec{v}$.
   c. Has unit length and is parallel to $\vec{v}$.

8. What would you get if you drew all possible unit vectors in 3-space with tails at the origin?

9. What object in three space is traced by the tips of all vectors starting at the origin that arc of the form
   a) $\vec{v} = \hat{i} + 2\hat{j} + b\hat{k}$ where $b$ is any real number.
   b) $\vec{v} = \hat{i} + a\hat{j} + b\hat{k}$ where $a$ and $b$ are any real numbers.

10. Describe the object created by all scalar multiples of $\vec{v} = \hat{i} - \hat{j}$ with tail at the point $(0, 0, 1)$.

11. Decide if each of the following statements is true or false.
   a) The length of the sum of two vectors is always strictly larger than the sum of the lengths of the two vectors.
   b) $\|\vec{v}\| = |v_1| + |v_2| + |v_3|$, where $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$.
   c) $\pm\hat{i}$, $\pm\hat{j}$, $\pm\hat{k}$ are the only unit vectors.
   d) $\vec{v}$ and $\vec{w}$ are parallel if $\vec{v} = \lambda\vec{w}$ for some scalar $\lambda$.
   e) Any two parallel vectors point in the same direction.
   f) $2\vec{v}$ has twice the magnitude as $\vec{v}$.

Answers Homework 13.1
1) $\langle 6, 11 \rangle$; 2) $\langle 8, 6 \rangle, \langle -8, -6 \rangle$; 3) $\pi + \tan^{-1}\left(\frac{3}{5}\right)$; 4) $(a = 2, b = -5)$; 5) $\frac{7\sqrt{9 - 5^2}}{\sqrt{106}}$;
6) $< -1, -1, 3 >$, $\sqrt{11}$; 7) a) $2<-1,2,7>$, b) $<-1,-2,-7>$, c) $2<-1,2,7>/\sqrt{54}$; 8) a unit sphere; 9) Line parallel to $z$-axis through point $(1,1)$, plane $x=1$; 10) Line through point $(0,0,1)$ parallel to $\vec{v} = \hat{i} - \hat{j}$; 11) Only d and f are true.
13.2 Vectors In General

eg 10 Give the components of the velocity vector of an airplane moving at 45mph in the direction 30° east of south.
The components are $< 45 \cos(60°), -45 \sin(60°) > = < 22.5, -22.5\sqrt{3} >$.

eg 11 An airplane with airspeed of 200 mph is headed N60°W. A wind is blowing directly south. If the true direction of the airplane is west, find the true speed of the airplane and the speed of the wind.
The true speed of the plane is the $x$-component $200\sin(60°) = 100\sqrt{3}$ mph.
The speed of the wind is the $y$-component $200\cos(60°) = 100$ mph.

eg 12 A ship is traveling at a speed of 22.0 knots at a compass heading of 95.0° (measured clockwise from the north). The current is flowing due south at a speed of 5.00 knots. Find the actual speed $v$ and compass heading $\theta$ of the ship.

If we use trigonometry, since the angle between the direction of the ship and the current is 95.0°, by law of cosines, the actual speed of the ship $|\vec{v}| = \sqrt{22^2 + 5^2 - 2 \cdot 22 \cdot 5 \cdot \cos(95°)} \approx 22.98 = 23.0$ knots to 3 sig. If we use the law of sines, $\frac{\sin(95°)}{22.98} = \frac{\sin(\alpha)}{22.0}$, giving $\alpha \approx 72.5°$, so the actual heading of the ship is 107.5°.

If we use vectors, the ship vector $\vec{s} = < 22.0\cos(5.0°), -22.0\sin(5.0°) >$ and the current vector $\vec{c} = < 0, -5.00 >$. If we add the two vectors, we have $\vec{s} + \vec{c} = < 22.0\cos(5°), -22.0\sin(5°) - 5.00 >$.
The magnitude of the vector 23.0 knots is the actual speed. The direction of the vector $\tan(\gamma) = \frac{y}{x}$, so $\gamma \approx -17.5°$ is the reference angle, so the actual heading of the ship is 107.5°.

eg 13 Find the resultant $\vec{F}$ force at a point, if the angle of $\vec{F}_1$ with magnitude 8 lb. is 135°, and the angle of $\vec{F}_2$ with magnitude 6 lb. is 30° at that point.
Since $\vec{F}_1 = < 8\cos\frac{3\pi}{4}, 8\sin\frac{3\pi}{4} >$; $\vec{F}_2 = < 6\cos\frac{\pi}{6}, 6\sin\frac{\pi}{6} >$,

$\vec{F}_1 + \vec{F}_2 = < 8\cos\frac{3\pi}{4} + 6\cos\frac{\pi}{6}, 8\sin\frac{3\pi}{4} + 6\sin\frac{\pi}{6} > =$

$<-4\sqrt{2} + 3\sqrt{3}, 4\sqrt{2} + 3 >$ with direction $\tan(\theta) = \frac{y}{x} = \frac{4\sqrt{2} + 3}{-4\sqrt{2} + 3\sqrt{3}}$, or $\theta \approx -87° + 180° = 93°$.
eg 14 An object is pulled by a force $\vec{F}$ in the direction of $< 1, 1 >$ and force $\vec{G}$ in the direction of $< 3, -2 >$. Find the two forces if the resultant force is $< 50, 0 >$.

Since $\vec{F} + \vec{G} = a < 1, 1 > + b < 3, -2 > = < 50, 0 > \Rightarrow a + 3b = 50$ and $a - 2b = 0$. By solving the system, $b = 10$ and $a = 20$, so $\vec{F} = 20 < 1, 1 >$ and $\vec{G} = 10 < 3, -2 >$.

eg 15 An object is pulled by the forces $\vec{F} = 2 \angle \frac{\pi}{3}$, $\vec{G} = 2\sqrt{2} \angle -\frac{\pi}{4}$ and $\vec{H} = 2 \angle \frac{\pi}{6}$.

Find the resultant force, with its magnitude and direction.

If we represent the angles as unit vectors, the resultant force will be

$$\vec{R} = 2 \frac{< 1, \sqrt{3} >}{2} + 2\sqrt{2} \frac{< 1, -1 >}{\sqrt{2}} + 2 \frac{< \sqrt{3}, 1 >}{2} = < 3 + \sqrt{3}, \sqrt{3} - 1 > \text{ with magnitude } \sqrt{16 + 4\sqrt{3}} \text{ and direction }$$

$$\theta = \tan^{-1} \frac{\sqrt{3} - 1}{3 + \sqrt{3}} \text{, since } \theta \text{ is in the first quadrant.}$$

eg 16 A 100 lb. weight hangs from two wires. The wire to the left has a tension $T_1$ and an angle of depression of $50^\circ$ and the one to the right has a tension $T_2$ and an angle of depression of $32^\circ$. Find the two tensions.

Let $T_1 = \left| \vec{T_1} \right|$ and $T_2 = \left| \vec{T_2} \right|$

$$\vec{T_1} = T_1 < -\cos(50^\circ), \sin(50^\circ) > \text{ and } \vec{T_2} = T_2 < \cos(32^\circ), \sin(32^\circ) >.$$ 

Since the two tensions counterbalance the weight $\vec{w} = < 0, -100 >$, the sum of the x-components is zero and the sum of the y-components is 100.

So; $-T_1 \cos(50^\circ) + T_2 \cos(32^\circ) = 0$ and $T_1 \sin(50^\circ) + T_2 \sin(32^\circ) = 100$.

By solving the system, $T_1 = \frac{100 \cos(32^\circ)}{\sin(82^\circ)} \approx 85.64 \text{lb}$ and $T_2 \frac{100 \cos(50^\circ)}{\sin(82^\circ)} \approx 64.91 \text{lb}$. We have used the identity $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$. 

7
Homework 13.2

1. A skater skates in the direction of the vector \(\langle 4, 3 \rangle\) from point P to point R and then in the direction of \(\langle -4, 2 \rangle\) to Q, where \(\overrightarrow{PQ} = \langle 0, 50 \rangle\) meters (figure below). How far is R from P and how far is R from Q?

![Diagram](image)

2. The force of the wind on a rubber raft is twenty-five pounds toward the southwest. What force must be exerted by the raft’s motor for the combined force of the wind and motor to be twenty pounds toward the south?

3. If weight of 500 lb is supported as shown in the figure, draw a force diagram to find the exact magnitude of the forces in the members CB and AB.

![Diagram](image)

4. A 75lb. weight is suspended from two wires. If \(\vec{F}_1\) makes an angle of 55° with the horizontal, and \(\vec{F}_2\) makes an angle of 40° with the horizontal, find the magnitude of the two forces.

5. An object is pulled by the forces \(\vec{F} = 2 \angle \frac{\pi}{3}\), \(\vec{G} = 2\sqrt{2} \angle -\frac{\pi}{4}\). Find the force that must be applied to the object if it is to remain stationary. Simplify your answer completely.
6. The boat in the figure below is being pulled into a dock by two ropes. The force \( \mathbf{F} \) by the upper rope has the direction of the vector \( \langle 2, 2 \rangle \) with the usual orientation of coordinate axes, and the force \( \mathbf{G} \) by the lower rope is in the direction of \( \langle 6, -4 \rangle \). The total force on the boat by the two ropes is \( \langle 100, 0 \rangle \) pounds. What are the magnitudes of the forces by the two ropes?

\[
\begin{align*}
\mathbf{F} & \text{ and } \mathbf{G}
\end{align*}
\]

7. Two wires suspend a twenty-five-pound weight. One wire makes an angle of 45° with the vertical and the tension in it (the magnitude of the force it exerts) is twenty pounds. What is the tension in the other rope and what angle does it make with the vertical? (Notice that the angle with the vertical is not the angle of inclination.)

8. A sailor tacks toward the northeast from point P to point R and then tacks toward the northwest to Q (figure below). The point Q is 300 meters north and 100 meters east of P. How far does the sailor travel on each of the tacks?

\[
\begin{align*}
\mathbf{F} & \text{ and } \mathbf{G}
\end{align*}
\]

9. The figure below shows four forces, measured in Newtons, that are applied to a ring. Find a magnitude and angle of inclination of their resultant.

\[
\begin{align*}
\mathbf{F} & \text{ and } \mathbf{G}
\end{align*}
\]

10. A plane is heading due east and climbing at the rate of 80 km/hr. If its airspeed is 480 km/hr. and there is a wind blowing 100 km/hr. to the northeast, what is the ground speed of the plane?

11. An airplane heads northeast at an airspeed of 700 km/hr, but there is a wind blowing from the west at 60 km/hr. In what direction does the plane end up flying? What is its speed relative to the ground? Ans: 48.3° relative to North, 744 km/hr.
Answers Homework 13.2

1) \[ |\overrightarrow{PR}| = 50, \quad |\overrightarrow{QR}| = 20\sqrt{5}; \quad 2) \left(\frac{25}{\sqrt{2}}, \frac{25}{\sqrt{2}} - 20\right); \quad 3) \text{CB: 1000lb, AB: 500 \sqrt{3} lb}; \quad 4) \bar{F}_1 = 57.67\text{lb}, \bar{F}_2 = 43.18\text{lb}; \quad 5) \langle -3, 2 \rangle, \sqrt{3}; \quad 6) \text{upper 40\sqrt{2}}, \text{lower 20\sqrt{13}}; \quad 7) 17.83 \text{ lb}; \quad 8) \langle 200, 2 \rangle, \langle 100, 2 \rangle; \quad 9) 17.3\text{ N}; \quad 10) 549 \text{ km/hr}; \quad 11) 48.3^\circ \text{ relative to North, 744 km/hr.}

13.3 The Dot Product

### Dot Product: (Scalar Product)

The dot product is defined geometrically by

\[ \overrightarrow{u} \cdot \overrightarrow{v} = |\overrightarrow{u}| |\overrightarrow{v}| \cos \theta, \quad u \neq 0, v \neq 0 \]

where \( \theta \) is the angle between the vectors \( \overrightarrow{u} \) and \( \overrightarrow{v} \) (0 \( \leq \theta \) \( \leq \pi \)).

If \( \overrightarrow{u} = \langle u_1, u_2 \rangle \) and \( \overrightarrow{v} = \langle v_1, v_2 \rangle \) then the dot product is defined in the Cartesian coordinate system by

\[ \overrightarrow{u} \cdot \overrightarrow{v} = u_1 v_1 + u_2 v_2. \]

**Proof:** Consider \( \overrightarrow{u} = \langle u_1, u_2 \rangle \) and \( \overrightarrow{v} = \langle v_1, v_2 \rangle \).

Since the vector \( \overrightarrow{u} - \overrightarrow{v} = \langle u_1 - v_1, u_2 - v_2 \rangle \), the magnitude squared of this vector will be \( |\overrightarrow{u} - \overrightarrow{v}|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2 \). This last expression can be expanded and rearranged as \( (u_1^2 + u_2^2) + (v_1^2 + v_2^2) - 2(u_1 v_1 + u_2 v_2) \), so \( |\overrightarrow{u} - \overrightarrow{v}|^2 = |\overrightarrow{u}|^2 + |\overrightarrow{v}|^2 - 2(u_1 v_1 + u_2 v_2) \).

If we use the law of cosines with \( \theta \) the angle between \( \overrightarrow{u} \) and \( \overrightarrow{v} \), we can say

\[ |\overrightarrow{u} - \overrightarrow{v}|^2 = |\overrightarrow{u}|^2 + |\overrightarrow{v}|^2 - 2 |\overrightarrow{u}| |\overrightarrow{v}| \cos (\theta). \]

By comparing the last two expressions we can conclude that

\[ |\overrightarrow{u}| |\overrightarrow{v}| \cos (\theta) = u_1 v_1 + u_2 v_2. \]

In space, if \( \overrightarrow{u} = \langle u_1, u_2, u_3 \rangle \) and \( \overrightarrow{v} = \langle v_1, v_2, v_3 \rangle \) then the dot product is defined in the Cartesian coordinate system by \( \overrightarrow{u} \cdot \overrightarrow{v} = u_1 v_1 + u_2 v_2 + u_3 v_3. \)

**eg 17** If \( \overrightarrow{u} = \langle -3, 0 \rangle \) and \( \overrightarrow{v} = \langle -1, -1 \rangle \) then \( \overrightarrow{u} \cdot \overrightarrow{v} = |\overrightarrow{u}| |\overrightarrow{v}| \cos \theta = (3)(\sqrt{2}) \cos \frac{\pi}{4} = 3; \) since \( \theta \) is \( \frac{\pi}{4} \).

Also, by using the definition, \( \overrightarrow{u} \cdot \overrightarrow{v} = (-3)(-1) + (0)(-1) = 3. \)

Note: The dot product gives a number.
Angle Between Two Vectors

The angle between two non-zero vectors in 2-space can be found by using the dot product.

\[ \cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}, \quad \theta = \begin{cases} \text{acute angle if } \vec{u} \cdot \vec{v} > 0 \\ \text{obtuse angle if } \vec{u} \cdot \vec{v} < 0 \\ \pi/2 \text{ if } \vec{u} \cdot \vec{v} = 0 \end{cases} \]

Two non-zero vectors are orthogonal (perpendicular) if \( \vec{u} \cdot \vec{v} = 0 \).
The vectors \( \hat{i}, \hat{j} \) and \( \hat{k} \) are orthogonal since \( \hat{i} \cdot \hat{j} = 0, \hat{j} \cdot \hat{k} = 0, \hat{k} \cdot \hat{i} = 0 \).

eg 18 Find the angle between the vectors \( <0, 1> \) and \( <-1, -\sqrt{3}> \).

\[ \cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u||v}|} = \frac{-\sqrt{3}}{2}, \theta = \frac{5\pi}{6}. \]

Properties of Dot Products:
For any vectors \( \vec{u}, \vec{v}, \vec{w} \) and any scalar \( k \),
a) \( \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \)
b) \( \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \)
c) \( k(\vec{u} \cdot \vec{v}) = (k \vec{u}) \cdot \vec{v} = \vec{u} \cdot (k \vec{v}) \)
d) \( \vec{v} \cdot \vec{v} = |\vec{v}||\vec{v}| = |\vec{v}|^2 \text{ or } |\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}} \)

Work
The work \( W \) done by a constant force \( \vec{F} \) in the direction of motion \( \vec{D} \) is given by \( W = \vec{F} \cdot \vec{D} = |\vec{F}| |\vec{D}| \cos \theta \) where \( \theta \) is the angle between the force and the direction of motion.

eg 19 Find the work needed to move an object 3m if a 6N force is applied to the object at an angle of \( \frac{\pi}{6} \) with the direction of motion.

\[ W = \vec{F} \cdot \vec{D} = (3 \cos \frac{\pi}{6}) 6 = \frac{9\sqrt{3}}{N} \text{ m} = 9\sqrt{3} \text{ J (joules)}. \]

or, \( <3, 0> \cdot <6 \cos (\pi/6), 6 \sin (\pi/6)> = 9\sqrt{3} \)

eg 20 A constant force with vector representation \( \vec{F} = i + 2j \) moves an object along a straight line from the point \( (2, 4) \) to the point \( (5, 7) \). Find the work done in foot-pounds if force is measured in pounds and distance is measured in feet. Ans. [9 ft-lb]
**Directional Cosines**

Finding angles between vectors in 3-space and the coordinate axis

Consider the unit vector \( \hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{\langle u_1, u_2, u_3 \rangle}{|\vec{u}|} = \langle \frac{u_1}{|\vec{u}|}, \frac{u_2}{|\vec{u}|}, \frac{u_3}{|\vec{u}|} \rangle \).

Any unit vector can be expressed in terms of its directional cosines \( \alpha, \beta, \gamma \), since \( \cos \alpha = \frac{\vec{u}_1}{|\vec{u}|}, \cos \beta = \frac{\vec{u}_2}{|\vec{u}|}, \cos \gamma = \frac{\vec{u}_3}{|\vec{u}|} \). From the components of any unit vector in 3-space, we can obtain the directional cosines where

\[
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{u_1^2}{|\vec{u}|^2} + \frac{u_2^2}{|\vec{u}|^2} + \frac{u_3^2}{|\vec{u}|^2} = \frac{u_1^2 + u_2^2 + u_3^2}{|\vec{u}|^2} = 1.
\]

\[\text{eg 21} \quad \text{Find the directional cosines of } \hat{u} = \langle 2, -1, -2 \rangle.\]

\[
\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{\langle 2, -1, -2 \rangle}{\|2, -1, -2\|} = \langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle. \quad \text{So } \cos \alpha = \frac{2}{\sqrt{5}}, \cos \beta = \frac{-1}{\sqrt{5}}, \cos \gamma = \frac{-2}{\sqrt{5}}.
\]

The directional cosines apply also to vectors in 2D. A unit vector in 2D can be represented as

\[\langle \cos \alpha, \cos \beta \rangle = \langle \cos \alpha, \cos (\pi/2 - \alpha) \rangle = \langle \cos \alpha, \sin \alpha \rangle \text{ since } \alpha \text{ and } \beta \text{ are complementary angles.} \]

**Projections**

The component of the vector \( \vec{b} \) along \( \hat{a} \) is \( |\vec{b}| \cos \theta \) where \( \theta \) is the angle between \( \vec{b} \) and \( \hat{a} \).

It is expressed as \( \text{comp}_a \vec{b} = \frac{|\vec{a}||\vec{b}| \cos \theta}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = |\vec{b}| \cos(\theta). \)

The projection (also called parallel projection) vector of \( \vec{b} \) along \( \hat{a} \) and is expressed as

\[\text{proj}_a \vec{b} = |\vec{b}| \cos \theta \hat{a} = \frac{|\vec{a}| |\vec{b}| \cos \theta}{|\vec{a}|} \hat{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \hat{a}.
\]

The projection vector of \( \vec{b} \) orthogonal to \( \hat{a} \) (also called perpendicular projection) is

\[\hat{c} = \text{orth}_a \vec{b} = \vec{b} - \text{proj}_a \vec{b} = \vec{b} - \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \hat{a}.
\]

\[\text{eg. 22} \quad \text{Let } \vec{b} = \langle 3, 2 \rangle; \hat{a} = \langle 2, 1 \rangle;\]

\[\text{comp}_a \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{8}{\sqrt{5}}; \text{proj}_a \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \hat{a} = \frac{8}{5} \langle 2, 1 \rangle;\]

\[\text{orth}_a \vec{b} = \langle 3, 2 \rangle - \frac{8}{5} \langle 2, 1 \rangle = \frac{1}{5} \langle -1, 2 \rangle.
\]

\[\text{eg. 23} \quad \text{Let } \vec{b} = \langle 3, 2, -1 \rangle; \hat{a} = \langle 2, 1, 1 \rangle;\]

\[\text{comp}_a \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{6+2-1}{\sqrt{6}} = \frac{7}{\sqrt{6}}.\]

\[\text{proj}_a \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \hat{a} = \frac{6+2-1}{6} \langle 2, 1, 1 \rangle = \frac{7}{3} \langle 2, 3, 6 \rangle;\]

\[\text{orth}_a \vec{b} = \vec{b} - \text{proj}_a \vec{b} = \langle 3, 2, -1 \rangle - \frac{7}{3} \langle 2, 3, 6 \rangle = \langle \frac{2}{3}, \frac{5}{6}, -\frac{13}{6} \rangle.
\]
Any vector can be expressed as the sum of its parallel and perpendicular projection to another vector. Since \( \text{proj}_a \overrightarrow{b} + \text{orth}_a \overrightarrow{b} = \text{proj}_a \overrightarrow{b} + (\overrightarrow{b} - \text{proj}_a \overrightarrow{b}) = \overrightarrow{b} \).

**eg. 24** Let \( \overrightarrow{b} = <3, 2> \); \( \overrightarrow{a} = <2, 1> \). Express \( \overrightarrow{b} \) as the sum of its parallel and perpendicular projection to \( \overrightarrow{a} \)

\[
\overrightarrow{b} = <3, 2> = \text{proj}_a \overrightarrow{b} + \text{orth}_a \overrightarrow{b} = \frac{8}{5} <2, 1> + \frac{1}{5} < -1, 2 >.
\]

**eg 25** Let \( \overrightarrow{b} = <3, 2, -1> \); \( \overrightarrow{a} = <2, 1, 1> \). Express \( \overrightarrow{b} \) as the sum of its parallel and perpendicular projection to \( \overrightarrow{a} \)

\[
\overrightarrow{b} = <3, 2, -1> = \text{proj}_a \overrightarrow{b} + \text{orth}_a \overrightarrow{b} = <\frac{7}{3}, \frac{7}{6}, \frac{7}{6}> + <\frac{2}{3}, \frac{5}{6}, \frac{-13}{6} >.
\]

The angle between the vectors is also given by \( \cos(\theta) = \frac{|\overrightarrow{a}||\overrightarrow{b}|\cos(\theta)}{|\overrightarrow{a}||\overrightarrow{b}|} = \frac{\text{comp}_a \overrightarrow{b}}{|\overrightarrow{a}|}. \)

**eg 26** The angle between \( \overrightarrow{b} = <3, 2> \) and \( \overrightarrow{a} = <2, 1, 1> \) is \( \cos(\theta) = \frac{\sqrt{21}}{5} \), or \( \theta \approx 7.12^\circ \).

**eg 27** The angle between \( \overrightarrow{b} = <3, 2, -1> \) and \( \overrightarrow{a} = <2, 1, 1> \) is \( \cos(\theta) = \frac{7}{\sqrt{84}} = \frac{7}{\sqrt{84}} \), or \( \theta \approx 40.2^\circ \).

**Planes**

The equation of a plane is determined by a point on the plane and a normal to the plane. Let \( \overrightarrow{r} = <x, y, z> \) be a position vector in space that describes a plane. If \( \overrightarrow{n} = <x_0, y_0, z_0> \) is the position vector of the point \((x_0, y_0, z_0)\) on the plane and \( \overrightarrow{n} = <a, b, c> \) is the normal to the plane, the equation of the plane will be given by \((\overrightarrow{r} - \overrightarrow{n}) \cdot \overrightarrow{n} = 0 \). So \( ax + by + cz = (ax_0 + by_0 + cz_0) = 0 \). We can write the equation as \( ax + by + cz + d = 0 \) where \( d = -(ax_0 - by_0 + cz_0) \).

**Eg28** Find the equation of the plane through \((2, 4, -1)\) with \( \overrightarrow{n} = <2, 3, 4> \).

\[2x + 3y + 4z - ((2)(2) + (3)(4) + (4)(-1)) = 0\]. So \(2x + 3y + 4z - 12 = 0\) is the equation of the plane.

**Eg29** Give the equation of the plane through \((1, -1, 2)\) parallel to the plane \(3x - 5y + 6z = 10\). Since the normal of both planes are the same, \( \overrightarrow{n} = <3, -5, 6> \), the equation of the parallel plane becomes \(3x - 5y + 6z = 3 + 5 + 12\) or \(3x - 5y + 6z = 20\).

**Perpendicular Distance Between a point and a Line in 2-space**

If \(p(x_1, y_1)\) is a fixed point, show that the minimum distance between the point and the line \(ax + by + c = 0\) is given by \(d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}\).

**Method 1.** Use Calculus to minimize the distance between the point and the line.

\[s = d^2 = (x - x_1)^2 + (y - y_1)^2\]. If we substitute the line \(y = \frac{-c-ax}{b}\) into the distance formula, we have
s = d^2 = (x - x_1)^2 + \left(\frac{-c-ax}{b} - y_1\right)^2. To minimize we need \( \frac{ds}{dx} = 0. \)

\[
\frac{ds}{dx} = 2(x - x_1) + 2 \left(\frac{-c-ax}{b} - y_1\right) \left(-\frac{a}{b}\right) = 0.
\]

Solving for x we have \( x = \frac{x_1 b^2 - ca - y_1 b a}{a^2 + b^2} \). If we substitute x and y into s, we obtain \( s = d^2 = \frac{(ax_1 + by_1 + c)^2}{a^2 + b^2} \) or \( d = \frac{\sqrt{(ax_1 + by_1 + c)^2}}{a^2 + b^2} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \).

**Method 2. Use Vectors**

First show that the vector \( < a, b > \) is perpendicular to the line \( ax + by + c = 0 \).

One way to show this is to let the points \((x_1, y_1)\) and \((x_2, y_2)\) be on the line. A vector on the line is given by \( \vec{v} = < x_2 - x_1, y_2 - y_1 > \). If we assume a normal vector \( \vec{n} = < a, b > \), then \( \vec{v} \cdot \vec{n} = a(x_2 - x_1) + b(y_2 - y_1) = (ax_2 + by_2) - (ax_1 + by_1) \). Since the two points satisfy the line, we have \( (ax_2 + by_2) - (ax_1 + by_1) = (c) - (c) = 0 \) thus \( \vec{v} \) and \( \vec{n} \) are perpendicular since the dot product is zero.

Another way is to use slopes. We can show that \( \vec{n} = < a, b > \), is perpendicular to the line \( ax + by + c = 0 \). Since the slope of the line is \( m = -\frac{a}{b} \), and \( m_{\text{per}} = \frac{b}{a} \), a vector perpendicular to the line will be \( < a, b > \).

Let \((x_0, y_0)\) be a point on the line and \((x_1, y_1)\) a fixed point. The distance between \((x_1, y_1)\) and the line will be given by \( D = |\text{comp} \rightarrow \vec{n}| = \frac{|\vec{r} \cdot \vec{n}|}{|\vec{n}|} = \frac{|a(x_1-x_0)+b(y_1-y_0)|}{\sqrt{a^2+b^2}} = \frac{|ax_1 + by_1 - ax_0 - by_0|}{\sqrt{a^2+b^2}} \) where \( \vec{n} = < a, b > \) and \( \vec{r} = < (x_1 - x_0), (y_1 - y_0) > \). Since the point \((x_0, y_0)\) satisfies the line then \( D = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} \).

**eg 30** Find the distance between the point \((1,3)\) and the line \(3x - 4y + 1 = 0\).

\[
D = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} = \frac{|3(1) - 4(3) + 1|}{\sqrt{3^2+4^2}} = \frac{8}{5}.
\]

**eg 31** Find the distance between the two lines \( l_1: (2x - 3y = 1) \) and \( l_2: (-4x + 6y = 1) \).

Since the two lines are parallel, we can choose any point in one of the lines, and a vector normal to the lines. A point in \( l_2 (x_1, y_1) = (0, 1/6) \). So \( D = \frac{|2(0) - 3(1/6) - 1|}{\sqrt{2^2+3^2}} = \frac{3/2}{\sqrt{13}} \).

**Parallel and Perpendicular Vectors to the Tangent Line of a Curve**

The parallel (tangent) vector to the tangent line of \( y = f(x) \) at \( x_0 \) can be found by finding the slope \( m_1 \) of the tangent line. The perpendicular (normal) vector can be found from the slope of the normal line \( m_2 \), since \( m_1 m_2 = -1 \) for perpendicular lines.

**eg 32** Find two unit tangent and two unit normal vectors of \( f(x) = \frac{x^3}{2} + \frac{1}{2} \) at \((1,1)\).

Since \( f'(x) = \frac{3x^2}{2} \bigg|_{x=1} = \frac{3}{2} = \frac{\Delta y}{\Delta x} \), the unit tangent vectors will be \( \pm <\frac{2}{\sqrt{13}}\> \), and the unit normal vectors
1. Give the two unit vectors in an $xy$-plane that are parallel to the line $y = 2x + 1$.

2. Give the two unit vectors in an $xy$-plane that are normal to $y = \sin x$ at $x = \pi / 3$.

3. Find numbers $A$ such that $\langle -4, 7 \rangle$ and $\langle A, 3 \rangle$ (a) are parallel and (b) are perpendicular.

4. (a) Calculate $A \cdot B$ for $A = i + 5j$ and $B = 6i + 4j$. (b) What is the component of $A$ in the direction of $B$? (c) Find the projection of $A$ on a line through $B$. (d) What is the length of $\text{proj}_A \vec{B}$? (e) Find $\text{orth}_A \vec{B}$ (f) What vector do you obtain when you add $\text{proj}_A \vec{B} + \text{orth}_A \vec{B}$?

5. Find exact and approximate decimal values of the angles (a) between $\langle 1, 3, 4 \rangle$ and $\langle -2, 0, 1 \rangle$ and (b) between $2i + 3j + 4k$ and $\text{-}3i + 4j - 5k$.

6. Find the direction cosines and the exact and approximate decimal values of the direction angles of $A = \langle 2, 1, -2 \rangle$.

7. Three ropes attached to a hook on it support a box. The ropes exert forces $F_1 = 20i + 10j + 15k$, $F_2 = -30i - 4j + 6k$, and $F_3 = 10i - 6j + 8k$ pounds, relative to $xyz$-coordinates with the positive $z$-axis pointing up. How much does the box weigh?

8. What is the component of $i + 2j - 3k$ in the direction from $(2, 5, 8)$ toward $(\text{-}2, 3, 9)$?

9. Find the constant $k$ such that the projection of $\langle 6, 10, k \rangle$ on a line parallel to $\langle -3, 2, 1 \rangle$ is $\langle 3 \cdot 2 \cdot 1 \rangle$.

10. Find the vector whose component in the direction of $i$ is 1, whose component in the direction of $j + k$ is $2\sqrt{2}$, and whose component in the direction of $i - 3j$ is $\sqrt{10}$.

11. Find a vector 6 unit long in the direction of $\vec{A} = \langle 2, 2, -1 \rangle$, whose component in the direction of $j + k$ is $2\sqrt{2}$, and whose component in the direction of $i - 3j$ is $\sqrt{10}$.

12. Show $\vec{u} = \langle 1, 0, 1 \rangle$ is perpendicular to $\vec{v} = \langle -1, 1, 1 \rangle$.

13. Find the directional cosines of $\langle 2, -1, -2 \rangle$.

14. Find the angle the position vector $\langle 1, 2, 3 \rangle$ makes with the $y$ axis.

15. Use the property of the dot product $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$ to derive a formula for the magnitude of the projection (component) of vector $\vec{u}$ into vector $\vec{v}$.

16. Find the component of $\vec{u} = \langle 1, 0, -1 \rangle$ parallel to $\vec{v} = \langle 1, 1, 1 \rangle$.

17. Write $\vec{A} = \langle 1,2,3 \rangle$ as the sum of a vector parallel to $\vec{B} = \langle 1,1,0 \rangle$ and a vector orthogonal to $\vec{B}$.

18. Write $\vec{A} = \langle 0,3,4 \rangle$ as the sum of a vector parallel to $\vec{B} = \langle 1,1,0 \rangle$ and a vector orthogonal to $\vec{B}$.

19. Write $\vec{B} = 1,2,3 \rangle$ as the sum of a vector parallel to $\vec{B} = \langle 3, -1, 0 \rangle$ and a vector orthogonal to $\vec{B}$.

20. Find the distance from the point $(2,4)$ to the line $y = -\frac{3}{4}x + 1$.

21. Use slopes to show that the vector $\langle 2, 3 \rangle$ is normal to the line $2x + 3y + 4 = 0$.

22. Use the dot product to show that the vector $\langle 3, -2 \rangle$ is normal to the line $3x - 2y + c = 0$. 

**Homework 13.3**
23. Find a unit tangent and a unit normal vector to the curves in 2-space.
   
   a) \( y = \cos(x) \) at \( x = \pi/3 \).
   
   b) \( y = \sin^{-1}(x) \) at \( x = \sqrt{2}/2 \).
   
   c) \( y = \tan^{-1}(x) \) at \( x = \sqrt{3}/3 \).

Answers Homework 13.3

1) ±(1, 2)/\( \sqrt{5} \); 2) ±(1, -2)/\( \sqrt{5} \); 3) a) A=12/7 b) A = -21/4; 4) a) 26, b) \( \sqrt{13} \), c) (1, 5), d) \( \sqrt{26} \), e) (5, -1), f) B; 5) \( \cos^{-1}(2/\sqrt{130}) \), \( \cos^{-1}(-14/\sqrt{1450}) \);

6) \( \alpha = \cos^{-1}\left(\frac{2}{3}\right), \beta = \cos^{-1}\left(-\frac{1}{3}\right), \gamma = \cos^{-1}\left(-\frac{2}{3}\right) \);

7) 29lb; 8) -11/\( \sqrt{21} \); 9) k = -4; 10) <1, -3, 7>;

11) 6 \( \vec{A} = \langle 4, 4, -2 \rangle \); 12) dot prod. =0; 13) \{\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\}; 14) \( \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \);

15) \( \text{comp} \to \vec{u} = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2} \); 16) \( \langle 2/\sqrt{3} \rangle \); 17) \( \vec{B} = \langle \frac{3}{2}, -\frac{1}{2}, -\frac{3}{2}, 3 \rangle \);

18) \( <\frac{3}{2}, \frac{3}{2}, 0, 0> + <\frac{3}{2}, \frac{1}{2}, 4, 4> \); 19) \( <\frac{3}{2}, -\frac{1}{2}, 0, 0> + <\frac{3}{2}, -\frac{3}{2}, -3, 3> \);

20) 18/5units;

23) a) ±\( \langle\frac{2, -\sqrt{3}, \sqrt{7}}{\sqrt{7}}\rangle \), ±\( \langle\frac{\sqrt{5}, 2, \sqrt{7}}{\sqrt{7}}\rangle \), b) ±\( \langle\frac{<\sqrt{2}, -1, \sqrt{3}>}{\sqrt{3}}\rangle \), ±\( \langle\frac{<\sqrt{2}, -1, \sqrt{3}>}{\sqrt{3}}\rangle \), ±\( \langle\frac{<4, 3, \sqrt{3}>}{5}\rangle \), ±\( \langle\frac{<3, -4, \sqrt{3}>}{5}\rangle \).

13.4 The Cross Product

Cross Product: (Vector Multiplication)

The cross product is an operation on two vectors in a three-dimensional space that results in another vector which is perpendicular to the plane containing the two input vectors. The direction of the new vector that results from the cross product of the vectors \( \vec{a} \) and \( \vec{b} \) is given by the right hand rule.

The cross product \( \vec{u} \times \vec{v} \) is defined:

\[
\vec{u} \times \vec{v} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{vmatrix}
= \hat{i} \left|\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\right| - \hat{j} \left|\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\right| + \hat{k} \left|\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right|
= \hat{i} (u_2v_3 - v_2u_3) - \hat{j} (u_1v_3 - v_1u_3) + \hat{k} (u_1v_2 - v_1u_2)
\]

We can see that the cross product is a vector. The direction of this vector is perpendicular to the plane that contains the vectors \( \vec{u} \) and \( \vec{v} \).

eg33. Find the cross product of \( \vec{u} = \langle 2, 1, 3 \rangle \) and \( \vec{v} = \langle 0, 2, 1 \rangle \).
\[ \hat{u} \times \hat{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 0 & 2 & 1 \end{vmatrix} = \hat{i} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = \langle -5, -2, 4 \rangle. \]

The magnitude of the cross product is given by \(|\hat{u} \times \hat{v}| = |\hat{u}| |\hat{v}| \sin(\theta)|, where \(\theta\) is the angle between \(\hat{u}\) and \(\hat{v}\), in the direction of the unit vector \(\hat{n}\) parallel to the vector \(\hat{u} \times \hat{v}\). The vector \(\hat{n}\) is perpendicular to the plane that contains the vectors \(\hat{u}\) and \(\hat{v}\).

Another way of writing the cross product vector is \(\hat{u} \times \hat{v} = |\hat{u}| |\hat{v}| \sin(\theta) \hat{n}\).

Two vectors are parallel if \(|\hat{u} \times \hat{v}| = 0\).

**Properties of Cross Products**

For any vectors \(\hat{u}, \hat{v}, \hat{w}\) in 3-space and any scalar \(k\),

a) \(\hat{u} \times \hat{v} = -\hat{v} \times \hat{u}\)

b) \(\hat{u} \times (\hat{v} + \hat{w}) = \hat{u} \times \hat{v} + \hat{u} \times \hat{w}\)

c) \(k (\hat{u} \times \hat{v}) = (k \hat{u} \times \hat{v}) = (\hat{u} \times k\hat{v})\)

d) \(\hat{v} \times \hat{v} = \hat{0}\)

1) \(\hat{u} \cdot \hat{u} \times \hat{v} = \hat{0}\) since \(\hat{u}\) and \(\hat{u} \times \hat{v}\) are perpendicular.

2) \(\hat{v} \cdot \hat{u} \times \hat{v} = \hat{0}\) since \(\hat{v}\) and \(\hat{u} \times \hat{v}\) are perpendicular.

**Areas**

The area of a parallelogram defined by the vectors \(\hat{u}\) and \(\hat{v}\) is given by \(A = \text{base} \times \text{height} = |\hat{u}| |\hat{v}| \sin(\theta) = |\hat{u} \times \hat{v}|\), where \(\theta\) is the angle between \(\hat{u}\) and \(\hat{v}\).

The area of a triangle defined by the vectors \(\hat{u}\) and \(\hat{v}\) is given by \(A = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2} |\hat{u}| |\hat{v}| \sin(\theta) = \frac{1}{2} |\hat{u} \times \hat{v}|\).

**eg35** Find the area of the parallelogram and the triangle defined by \(\hat{u} = \langle 1, 0, 1 \rangle\) and \(\hat{v} = \langle 2, 1, 0 \rangle\).

\[ \hat{u} \times \hat{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = \langle -1, 2, 1 \rangle. \] So the area of the parallelogram is \(|\hat{u} \times \hat{v}| = \sqrt{6} \text{ sq units}\) and the area of the triangle is \(\frac{\sqrt{6}}{2} \text{ sq units}\).

The cross product of the unit vectors are \(\hat{i} \times \hat{j} = \hat{k}; \hat{j} \times \hat{k} = \hat{i}; \hat{k} \times \hat{i} = \hat{j}\).

**eg 36** \(\hat{i} \times \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0, 0, 1 \rangle = \hat{k}\).
**Triple Scalar Product**

The triple scalar product is defined as 

\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \]

Proof:

\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \]

\[ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - c_1 b_3) + a_3 (b_1 c_2 - c_1 b_2) \]

Since interchanging two rows of a determinant only changes the sign of the determinant, it can be shown that 

\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) . \]

**Volume of a Parallelepiped**

The volume of a parallelepiped formed by the vectors \( \mathbf{a}, \mathbf{b} and \mathbf{c} \) is given by 

\[ |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \cos(\theta) \] where \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \times \mathbf{c} \).

If \( |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = 0 \), \( \mathbf{a} \) and \( \mathbf{b} \times \mathbf{c} \) are perpendicular so \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) are coplanar.

**eg37** Show that the vectors \( \mathbf{a} = <1, 4, -7>, \mathbf{b} = <2, -1, 4> \) and \( \mathbf{c} = <0, -9, 18> \) are coplanar.

\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1(18) - 4(36) - 7(-18) = 0. \]

The volume of the tetrahedron formed by the vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) is given by 

\[ \frac{1}{6} |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \]

**Planes**

Recall from a previous section that the equation of a plane is determined by a point on the plane and a normal to the plane.

Let \( \mathbf{r} = <x, y, z> \) be a position vector in space that describes a plane. If \( \mathbf{r}_0 = <x_0, y_0, z_0> \) is the position vector of the point \( (x_0, y_0, z_0) \) on the plane and \( \mathbf{n} = <a, b, c> \) is the normal to the plane, the equation of the plane will be given by \( (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0. \) So \( a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \) or \( ax + by + cz - (ax_0 + by_0 + cz_0) = 0. \) We can write the equation as \( ax + by + cz + d = 0 \) where \( d = -(ax_0 - by_0 + cz_0). \)
eg38 Find the equation of the plane that contains the points (1, 3, 2), (3, -1, 6), (5, 2, 0).

We can make the vectors \( \vec{u} = < 2, -4, 4 > \) from the first two points and \( \vec{v} = < 2, 3, -6 > \) from the last two.

Since the normal to the plane is \( \vec{n} = \vec{u} \times \vec{v} = < 12, 20, 14 > = < 6, 10, 7 > \), the plane is \( 6x + 10y + 7z - (6 + 30 + 14) = 0 \) or \( 6x + 10y + 7z - 50 = 0 \), where the point (1, 3, 2) was used as the point on the plane.

eg39 Find the smallest angle between the planes \( x + y + z = 1 \) and \( x - 2y + 3z = 1 \).

The angle between two planes will be the angle between their normal vectors \( \vec{R_1} = < 1, 1, 1 > \) and \( \vec{R_2} = < 1, -2, 3 > \).

By using the dot product, 
\[
\cos(\theta) = \frac{\vec{R_1} \cdot \vec{R_2}}{|\vec{R_1}| |\vec{R_2}|} = \frac{1-2+3}{\sqrt{3} \sqrt{14}} = \frac{2}{\sqrt{42}}
\]

so \( \theta \approx 72^\circ \).

eg40 Find a vector parallel to the intersection between the planes \( x + y + z = 1 \) and \( x - 2y + 3z = 1 \).

A parallel vector will be the cross product of the normal of each plane, so \( \vec{v} = < 5, -2, -3 > \).

Distance From a Point to a Plane

Let \( P_1 \) be the point \((x_1, y_1, z_1)\), and \( P_0 = (x_0, y_0, z_0) \) be any point on the plane \( ax + by + cz + d = 0 \). The distance \( D \) from \( P_1 \) to the plane will be \( D = |\text{comp} \vec{n} \cdot \vec{u}| = \frac{|\vec{n} \cdot \vec{u}|}{|\vec{n}|} \) where \( \vec{u} = < x_1 - x_0, y_1 - y_0, z_1 - z_0 > \), and \( \vec{n} = < a, b, c > \) is the normal to the plane.

\[
D = \frac{|\vec{n} \cdot \vec{u}|}{|\vec{n}|} = \frac{|ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}
\]

Since the point \((x_0, y_0, z_0)\) satisfies the plane \( ax + by + cz + d = 0 \), \( d = -(ax_0 + by_0 + cz_0) \), and

\[
D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}
\]

eg42 Find the distance from the point \( P_1 (2, -3, 4) \) to the plane \( x + 2y + 2z = 13 \).

\[
D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|1(2) + 2(-3) + 2(4) - 13|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{|-9|}{3} = 3
\]

eg41 Find the distance \( D \) from the point \( P_1 (2, 1, 5) \) to the plane that contains the point \( P_0 = (1, -1, 4) \) with normal \( \vec{n} = < 2, 4, 1 > \).

Let \( \vec{u} \) be the vector from \( P_0 \) to \( P_1 \). \( d \) is the \( |\text{comp} \vec{n} \cdot \vec{u}| \)

\[
D = \frac{|\vec{n} \cdot \vec{u}|}{|\vec{n}|} = \frac{|<2,4,1> \cdot <2,-1,1> + 5|}{\sqrt{2^2 + 4^2 + 1^2}} = \frac{|<2,4,1> \cdot <2,1,2>|}{\sqrt{21}} = \frac{11}{\sqrt{21}}.
\]
Homework 13.4

1. Find exact and approximate decimal values of the angles (a) between \( \langle 1, 3, 4 \rangle \) and \( \langle -2, 0, 1 \rangle \) and (b) between \( 2i + 3j + 4k \) and \(-3i + 4j - 5k\).

2. Three ropes attached to a hook on it support a box. The ropes exert forces \( F_1 = 20i + 10j + 15k \) lb., \( F_2 = -30i - 4j + 6k \) lb., and \( F_3 = 10i - 6j + 8k \) lb., relative to \( \text{xyz} \)-coordinates with the positive \( z \)-axis pointing up. How much does the box weigh?

3. What is the component of \( i + 2j - 3k \) in the direction from \((2, 5, 8) \) toward \((-2, 3, 9)\)?

4. Find numbers \( A \) such that \( \langle -4, 7 \rangle \) and \( \langle A, -3 \rangle \) (a) are parallel and (b) are perpendicular.

5. Find the constant \( k \) such that the projection of \( \langle 6, 10, k \rangle \) on a line parallel to \( \langle -3, 2, 1 \rangle \) is \( \left\langle \frac{3}{7}, \frac{2}{7}, \frac{1}{7} \right\rangle \).

6. Find the vector whose component in the direction of \( i \) is 1, whose component in the direction of \( j + k \) is \( 2\sqrt{2} \), and whose component in the direction of \( i - 3j \) is \( \sqrt{10} \).

7. Find a unit vector orthogonal to both \( \vec{u} = \langle 1, 0, 1 \rangle \) and \( \vec{v} = \langle -1, 1, 1 \rangle \).

8. For which values of \( t \) are the vectors \( \vec{A} = \langle t + 2, t, t \rangle \) and \( \vec{B} = \langle t - 2, t + 1, t \rangle \) are orthogonal?

9. Find the area of the triangle with vertices \( P(1,-1,0), Q(2,1,-1), R(-1,1,2) \).

10. Find the volume of the parallelepiped determined by \( \vec{A} = \langle t, -2, 0, 3 \rangle \) and \( \vec{C} = \langle 0, t, 7, -4 \rangle \).

11. Consider the parallelepiped with sides \( \vec{u} = \langle 1, 1, 1 \rangle \), \( \vec{v} = \langle -1, 1, 1 \rangle \), \( \vec{w} = \langle -1, 0, 1 \rangle \). Find the angle between the plane containing the vectors \( \vec{u} \) and \( \vec{v} \), and the vector \( \vec{w} \).

12. Find the distance from the point \( S(1,1,3) \) to the plane \( 3x + 2y + 6z = 6 \).

13. Find the distance from the point \( (2, -3, 4) \) to the plane \( x + 2y + 2z = 6 \).

14. Find the distance from the point \( (1, 1, 5) \) to the plane \( x + 2y + z = 4 \).

15. Find an equation for the plane through \( (0, 1, 0) \) and parallel to \( \vec{i} + \vec{j} \) and to \( \vec{j} - \vec{k} \).

16. Find an equation for the plane through \( (5, -1, -2) \) and perpendicular to the planes \( y - z = 4 \) and \( x + z = 3 \).

17. Find an equation for the plane through \( (2, 2, 4), (5, 6, 4), \) and \( (1, 3, 5) \).

18. Find the equation of the plane that contains the points \( (2,1,0), (1,2,1) \) and \( (3,-1,1) \).
19. Find an equation for the plane through \((1, 2, 3)\) and parallel to the plane \(4x - y + 3z = 0\).

20. Find a vector parallel to the intersection of the planes \(x + y + z = -1\) and \(x + 2y - z = -2\).

**Answers Homework 13.3**
1) a) \(\cos^{-1}(2/\sqrt{130})\), b) \(\cos^{-1}(-14/\sqrt{1450})\); 2) 29lb; 3) \(-11/\sqrt{21}\);
4) a) \(A=12/7\) b) \(A = -21/4\); 5) \(k = -4\); 6) \(<1, -3, 7>\); 7) \(\frac{\mathbf{u} \times \mathbf{v}}{||\mathbf{u} \times \mathbf{v}||} = <-1, -2, -1>/\sqrt{6}\); 8) \(t = \left\{\frac{4}{3}, 1\right\}\); 9) \(3\sqrt{2}\);
10) 23 cu-units; 11) \(\pi/6\); 12) \(\frac{17}{7}\); 13) \(\frac{2}{3}\); 14) \(\frac{2\sqrt{10}}{3}\); 15) \(x - y - z = -1\); 16) \(x - y - z = 8\); 17) \(4x - 3y + 7z = 30\); 18) \(3x + 2y + z = 8\); 19) \(4x - y + 3z = 11\); 20) \(<-1, 0, 1>\).