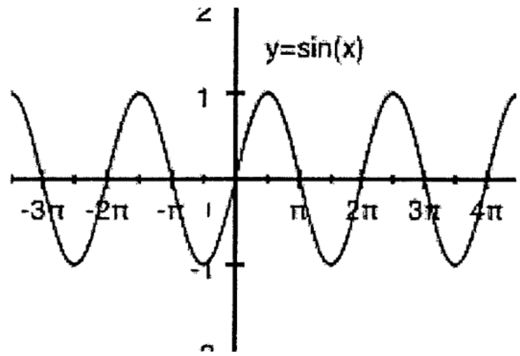
Solving Trig Equations Supplement

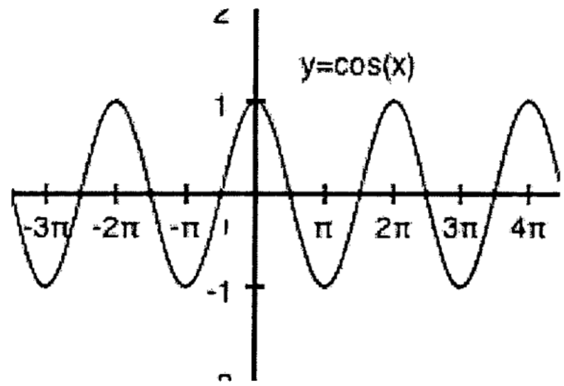
Section 1

In algebra class you spent a lot of time learning to solve equations. For example 2x + 1 = 15 which has the solution x = 7, and x^2 +1 = 10 which has two solutions x = 3 and x = -3. Keep in mind that we are finding values of x that will satisfy the equation. In this unit we will learn to solve equations where the unknown is "inside" a trig function. For example we will learn to solve sin(2x + 1) = 0. The key to solving equations like this is to use the graph of sine or cosine to get started.

The figure to the right shows the graph of y = sin(x). When we solve the equation sin(x) = 0 we are finding values of x for which sin(x) is zero. On the graph this means values of x where the y-coordinate is zero. We don't have to do any work. All we do is read the x values from the graph. Therefore six solutions to sin(x) = 0 are $x = -2\pi, -\pi, 0, \pi, 2\pi, 3\pi$ and we could continue this pattern to get as many solutions as desired.



Suppose we want to solve the equation cos(x) = 1. We would look at the graph of y=cos(x) and read the x values that have a y coordinate equal to 1. So the four solutions are $x = -2\pi, 0, 2\pi, 4\pi$ and infinitely many more following the same pattern.



Exercises:

1. Give three solutions to sin(x) = 1.

2. Give three solutions to cos(x) = 0.

3. Give five solutions to sin(x) = -1.

4. Give four solutions to cos(x) = -1.

Most of the trig equations we will encounter in applications are not quite as simple as those shown above. But solving those will be the basis for solving more complicated equations. For example suppose we want to solve the equation sin(2x — 1) = 0. The key here is to recognize that the equation has the form sine of "something" equals 0, or in symbols sin(θ) = 0. We can use the graph of sine to solve sin(θ) = 0 and get

$$\theta = 0 \quad \text{or} \quad \theta = \pi \quad \text{or} \quad \theta = 2\pi \quad \text{or} \quad \theta = 3\pi \quad \text{etc}$$

In the original equation sin(2x – 1) = 0 we have $\theta = 2x - 1$. Therefore from the line above we can write

$$2x - 1 = 0 \qquad 2x - 1 = \pi \qquad 2x - 1 = 2\pi \qquad 2x - 1 = 3\pi$$

which we can solve to get

$$2x - 1 \qquad 2x = \pi + 1 \qquad 2x = 2\pi + 1 \qquad 2x = 3\pi + 1$$

$$x = \frac{1}{2} \qquad x = \frac{\pi+1}{2} \qquad x = \frac{2\pi+1}{2} \qquad x = \frac{3\pi+1}{2}$$

We believe that these are solutions to sin(2x – 1) = 0. Recall from algebra that you can check a solution by substituting the value back into the equation and see if the equation is satisfied. For example to check the solution $x = \frac{1}{2}$ above, we would substitute $\frac{1}{2}$ for x in on the left side of the equation sin(2x – 1) = 0 and we get $\sin 2\left(\frac{1}{2}\right) - 1 = \sin(1 - 1) = \sin(0) = 0$ which means that $x = \frac{1}{2}$ is a solution. To check $x = \frac{\pi+1}{2}$ we would calculate $\sin\left(2\left(\frac{\pi+1}{2}\right) - 1\right) = \sin\big((\pi + 1) - 1\big) = \sin(\pi) = 0$.

Another example: Find three positive solutions to $\cos(\pi x) = 1$

Considering the graph of cosine we find that $\cos\theta = 1$ has solutions

$$\theta = 2\pi \text{ or} \qquad \theta = 4\pi \text{ or} \qquad \theta = 6\pi \text{ etc.}$$

Since the original equation is $\cos(\pi x) = 0$ we let $\theta = \pi x$ and use the line above to get

$$\pi x = 2\pi \text{ or} \qquad \pi x = 4\pi \text{ or} \qquad \pi x = 6\pi.$$

Solving for x we get

x = 2 or   x = 4 or   x = 6.

Sometimes we have to use algebra first to put the equation in the forms shown above. For example to solve $10\cos(\pi x) + 3 = 13$ we first solve for the trig function $\cos(\pi x)$. This will give

$$10\cos(\pi x) + 3 = 13$$

$$10\cos(\pi x) = 13 - 3$$

$$10\cos(\pi x) = 10$$

$$\cos(\pi x) = \frac{10}{10}$$

$$\cos(\pi x) = 1$$

At this point we have the same equation solved above and we proceed as before.

Exercises:

5. Find 3 solutions to $\cos(\pi x + 2) = 0$. Check your answer.

6. Find 4 solutions to $\sin(2x - 3) = 1$.

7. Find three solutions to $\cos(\pi x + 2) = -1$.

8. Find two solutions to $3\cos(2x) = 0$.

9. Find three solutions to $5\sin(3x) + 3 = 3$. (Hint: first solve for $\sin(3x)$)

10. Find three solutions to $3\sin(2x) + 5 = 2$.

11. The depth, D, of water at a boat dock is affected by tides and is given by

$D = 5 + 2\cos\left(\frac{\pi t}{6}\right)$ where *t* is the number of hours since high tide.

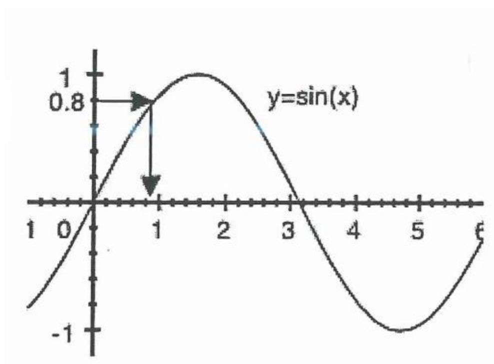
A. How long after high tide will the depth be 5 feet?

B. How long after high tide will the depth be 3 feet?

C. How long after high tide will the depth be 6 feet?

12. The electrical current at a point in a circuit is $C(t) = 20\sin(30\pi t)$ for $0 \leq t$ s. Find the first three times when the current is 0.

More Equation Solving



All of the equations we have solved so far reduced to the form $\sin(\theta) = 0$ or ± 1 or $\cos(\theta) = 0$ or ± 1. In this section we will see a more general procedure. Consider solving the equation $\sin(x) = 0.8$. By looking at the graph to the right we can see that there is a solution at approximately x = 0.9. Using the intersect command we could get a very accurate solution to the equation but since this type of equation is solved so often there is a function on the calculator that does the job. The function is called the *inverse sine* function and is denoted by $\sin^{-1}$. This function "undoes" the sine function. Specifically $\sin^{-1}(\sin(x)) = x$. To use this to solve the equation $\sin(x) = 0.8$, we will take inverse sine of both sides of the equation. This gives

$$\sin(x) = 0.8$$

Taking inverse sine gives

$$\sin^{-1}(\sin(x)) = \sin^{-1}(0.8)$$

$$x = \sin^{-1}(0.8) = 0.9273$$

Note that this is a reasonable answer based on the graph.

Another example: Solve the equation $5\cos(3x) = 2$.

First solve for $\cos(3x)$ by dividing both sides of the equation by 5.

$$\cos(3x) = \frac{2}{5}$$

Next "undo" the cos( ) by taking inverse cosine of both sides.

$$\cos^{-1}\big((3x)\big) = \cos^{-1}\left(\frac{2}{5}\right)$$

$$3x = \cos^{-1}\left(\frac{2}{5}\right)$$

Divide both sides by 3 to get

$$x = \frac{\cos^{-1}\left(\frac{2}{5}\right)}{3} \approx 0.386$$

Check this solution by substituting the value back into the equation.

Exercises

Find one solution to each of the equations. Check your answer.

13. $\sin(x) = .7$

14. $3\cos(x) = -2$

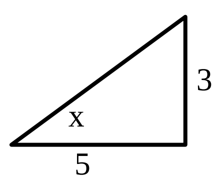
15. $5\sin(3x) = 1$

16. $4\cos(5x + 2) = 1$

17. $3\sin(\pi x + 2) + 5 = 7$

18. $\tan(3x) = 100$

19. Find the angle x in the right triangle shown.



20. The electrical current at a point in a circuit is $C(t) = 20\sin(30\pi t)$ for $0 \le t$. Find the first time when the current is 12.

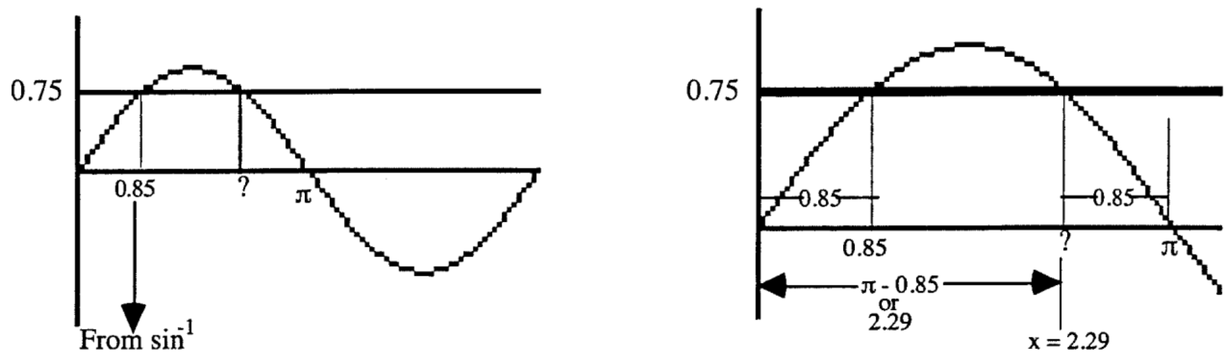
21. Calculate $\sin^{-1}(\sin(x))$ for x = 0.5, 1, 1.5, 2, 2.5, 3. This shows that the inverse sine does not "undo" the sine for all values of x. Based on the graph and the values calculated, what would you speculate is the largest value of x for which $\sin^{-1}(\sin(x)) = x$?

22. Calculate $\cos^{-1}(\cos(x))$ for x = 0, 1, 2, 3, 4, 5. This shows that the inverse cosine does not "undo" the cosine for all values of x. Based on the graph and the values calculated, what would you speculate is the largest value of x for which $\cos^{-1}(\cos(x)) = x$?

Section 2

In the previous section we learned to use inverse sine to get one solution to an equation like $\sin(x) = 0.75$. The solution is $x = \sin^{-1}(0.75) \approx 0.85$. What about other solutions?

The question we must answer is what is the other solution in the interval from 0 to 2π that the calculator did not give us. See the graph on the left below. The graph on the right shows a closer view of this graph. The space from the y-axis to the first solution is 0.85 and since sine is symmetric on the interval 0 to π, the space between π and the unknown value of x must be 0.85 too. Therefore the distance from the origin to the unknown value of x must be $\pi - 0.85$ or 2.29, which gives us the 2nd solution.



To find more solutions, keep adding 2π to each of the solutions $x \approx 0.85$ and $x \approx 2.29$. So two more solutions would be $x \approx 0.85 + 2\pi \approx 7.13$ and $x = 2.29 + 2\pi \approx 8.57$. Repeat this process of adding 2π to generate more solutions.

Example: Solve $\sin(5x) = 0.75$?

> Follow the approach shown earlier. Replace the "inside" part with θ. We just solved $\sin(x) = 0.75$ to find $x \approx 0.85$. It is the same for $\sin(\theta) = 0.75$. We know that $\theta \approx 0.85$. But instead of θ , we have 5x. So we have $5x \approx 0.85$ and $5x \approx 2.29$. Solving for x we get $x \approx 0.17$ and $x \approx 0.46$.

Another Example: Solve $\sin(5x + 1) = 0.75$

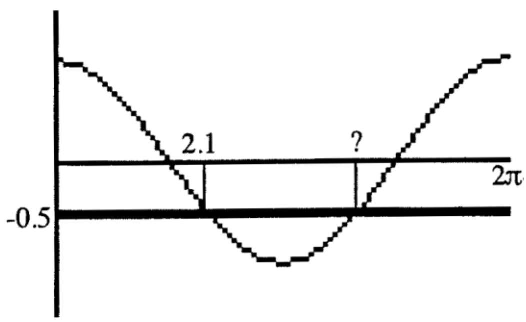
> Same idea. $\sin(\theta) = 0.75$, so $\theta_1 \approx 0.85$ (from $\sin^{-1}$) and $\theta_2 \approx 2.29$ (from the above procedure with graph at $\pi - 0.85$) But recall that $\theta = 5x + 1$.
>
> So, $5x + 1 \approx 0.85$ and $5x + 1 \approx 2.29$. Solve each for x. Gives $x \approx -0.03$ and 0.26.
>
> To get two more solutions we can add 2π to the solutions we get from $\sin(\theta) = 0.75$ and solve again for x. That is $5x + 1 \approx 0.85 + 2\pi$ and $5x + 1 \approx 2.29 + 2\pi$. Solve for x to get $x \approx 1.23$ and $x \approx 1.52$
>
> Note: To solve the equation $4\sin(5x + 1) - 3 = 0$ we start by solving for $\sin(5x + 1)$ by first adding 3 to both sides and then dividing both sides by 4. This gives $\sin(5x + 1) = 0.75$ and so we are back to the previous example.

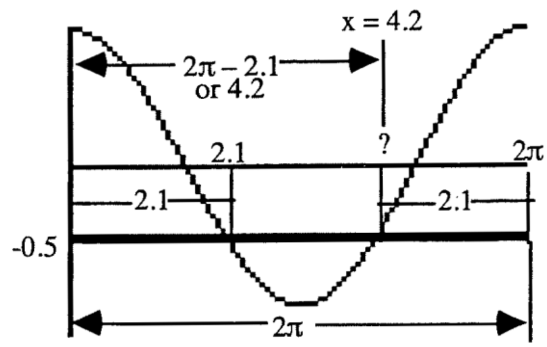
Class Exercise: Find two solutions to $3\sin(x+1) = 2$.

1. First solve for the sine. (Divide both sides by 3)
2. Replace the inside part(x + 1) with θ.
3. Solve $\sin\theta = \frac{2}{3}$ by using $\sin^{-1}$. That's one solution.
4. Find a second solution by thinking of the graph and symmetry.(π - solution from step 3)
5. Now that you have 2 solutions for θ in $\sin(\theta) = \frac{2}{3}$. Replace θ by x + 1 and solve for x.
6. Check your answer by substituting back into $3\sin(x+1) = 2$.

   Note that once the two solutions are found in step 3 and 4, you can get as many additional solutions as you like by adding multiples of 2π to the two we have.

What happens with cosine? Suppose we want to solve cos(x) = – 0.5. We get one solution from $\cos^{-1}(-0.5) = 2.1$ What about other solutions?





This time the space from the y-axis to the first solution is 2.1. Because of symmetry, the space from the unknown second solution to 2π is also 2.1. So the distance from the y-axis to the unknown second solution is $2\pi - 2.1$ or 4.2. So the other solution to the equation $\cos x = 0.5$ over the interval from 0 to 2π is $x = 4.2$. To generate more solutions add 2π to each solution as mentioned above.

Example: Find two solutions to $5\cos(4\mathrm{x}-3)=3$.

First solve for the cos(4x — 3).

$$\begin{aligned} 5\cos(4\mathrm{x}-3) &= 3 \\ \cos(4\mathrm{x}-3) &= 0.6 && \text{*Divide both sides by 5*} \\ \cos(\theta) &= 0.6 && \text{*Replace } 4x-3 \text{ with } \theta\text{*} \\ \theta &= \cos^{-1}(0.6) && \text{*Use } \cos^{-1} \text{ to solve*} \\ \theta_1 &= 0.93 \\ \theta_2 &= 2\pi - 0.93 && \text{*Use symmetry to find 2nd solution*} \\ \theta_2 &= 5.4 \end{aligned}$$

$$\begin{array}{rcl} \theta_1 = 0.93 & \theta_2 = 5.4 & \text{*Replace } \theta \text{ by } 4x-3 \text{ (the inside part)} \\ 4\mathrm{x}-3 = 0.93 & 4\mathrm{x}-3 = 5.4 & \text{*Solve each for x*} \\ 4\mathrm{x} = 3.93 & 4\mathrm{x} = 8.4 & \\ \mathrm{x} = 0.98 & \mathrm{x} = 2.1 & \end{array}$$

To generate more solutions add (or subtract) 2π to each solution for θ and then solve again for x. Note that additional solutions are always found for θ (not x) and then you re-solve the equation to get the x solutions.

Exercises:

23. Find the first two positive solutions to $\sin(\mathrm{x})=0.2$.

24. Find the first two positive solutions to $\sin(4\mathrm{x})=0.6$.

25. Find the first two positive solutions to $5\cos(\mathrm{x})=3$.

26. Find the first two positive solutions to $5\cos(2\mathrm{x})+1=4$.

27. Find four solutions to $3\sin(5\mathrm{x})=10$.

28. Find four solutions to $20\cos(\pi\mathrm{x})=7$.

29. Suppose that a 10 foot ladder is leaning against a house and the base of the ladder is 4 feet from the wall. What angle does the ladder make with the ground? Give the answer rounded to the nearest degree.

30. A ramp to provide wheelchair access to a building is 25 feet long and it rises up to a doorway that is 2 feet above the ground. What angle does the ramp make with the ground? Give the answer rounded to the nearest degree.

31. A boat leaves a dock and travels 5 miles due east of the dock and then turns and travels 3 miles north. A boat captain wants to tell a friend at the dock how to travel directly to the boat. How far is the boat from the dock and what angle does the direct line from the dock to the boat make with the east direction?

32. The line with the equation y = 2x makes what angle with the positive x-axis?

33. The line with the equation y = -3x makes what angle with the positive x-axis?

34. Draw a right triangle and label the sides so that $\sin(\theta) = \frac{2}{3}$. What is the exact value of the angle θ? What is the $\cos(\theta)$?

35. What angle with the positive x-axis does the line from the origin to the point (2, 5) make? Give the exact value and the decimal approximation rounded to two decimal places.

36. What angle with the positive x-axis does the line from the origin to the point (-4, 3) make? Give the exact value and the decimal approximation rounded to two decimal places.

Answers:

29. $\approx 66^{o}$

30. $\approx 4.6^{o}$

31. ≈ 5.83 miles and $\approx 31^{o}$

32. 63.4^{o} or 1.11 radians

33. 1.89 radians or 108^{o}

34. $\theta = \sin^{-1}\left(\frac{2}{3}\right)$ and $\cos\theta = \frac{\sqrt{5}}{3}$

35. $\tan^{-1}\left(\frac{5}{2}\right) \approx 68^{o}$

36. $\tan^{-1}\left(\frac{3}{-4}\right) \approx 143^{o}$