Parameterization and Vector Fields

17.1 Parameterized Curves

Curves in 2 and 3-space can be represented by parametric equations. Parametric equations have the form $x = x(t)$, $y = y(t)$ in the plane and $x = x(t)$, $y = y(t)$, $z = z(t)$ in space. Motion in the plane and space can also be described by parametric equations. To plot parametric curves in 3-space go to
http://cose.math.bas.bg/webMathematica/MSP/Sci_Visualization/3DParametricPlot

Review:

Parametric Curves in 2-space. To plot parametric curves in 2-space go to
http://cose.math.bas.bg/webMathematica/MSP/Sci_Visualization/ParametricPlot

eg 1 Show that the parametric equation $x = a\cos(t)$, $y = a\sin(t)$, $0 \le t \le 2\pi$ describes a circle or radius a.
We eliminate the parameter t by squaring x and y and adding. So $x^2 = a^2\cos^2(t)$, $y^2 = a^2\sin^2(t)$, or $x^2 + y^2 = a^2$ is an equation of a circle of radius a.

Parametric equations have orientation, and the parameterization of curves is not unique.

eg 2 Graph:

a) $x = \cos(t)$, $y = 4\sin^2(t)$, $0 \le t < \pi$;

b) $x = \cos(t)$, $y = 4\sin^2(t)$, $\pi \le t < 2\pi$;

c) $x = \frac{t}{2}$, $y = 4 - t^2$, $-2 \le t \le 2$.

a) If we eliminate the parameter t, $x^2 + \frac{y}{4} = 1$ or $x^2 = -\frac{1}{4}(y - 4)$. This is a segment of the parabola with vertex (0, 4) that opens down above the x-axis with a counterclockwise orientation.
i.e. From the point(1, 0) for $t = 0$ to the point (– 1, 0) for $t = \pi$.

b) This is the same parabola with clockwise orientation.
i.e. from the point (– 1, 0) for $t = \pi$ to the point (1, 0) for $t = 2\pi$

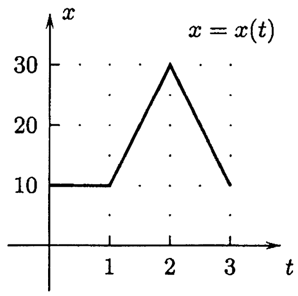
c) If we eliminate the parameter t, $y = 4 - 4x^2$. This is the same parabola with a different parameterization with clockwise orientation.
i.e. From the point (– 1, 0) for $t = -2$ to the point (1, 0) for $t = 2$.

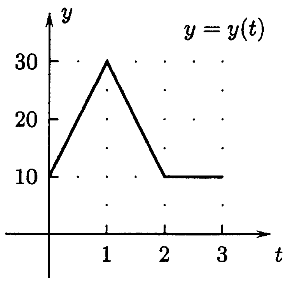
eg 3 Graph $x = b\cos(t)$, $y = a\sin(t)$, $0 \le t \le 2\pi$, $a \ge b$
This is an ellipse with a counterclockwise orientation with major axis of $2a$ on the *y-axis*.

eg 4 $x = t$, $y = t^2$, $z = 2$ $-\infty < t < \infty$ is a parabola on the plane $z = 2$.

eg 5 $x = \cos(t)$, $y = \sin(t)$, $z = t$ is a spiral of radius 1.

eg 6	Draw the curve whose parametric equations are given below.





If we make a table of points, we find the curve is a triangle with vertices (10, 10), (10, 30) and (30, 10) traversed in the CW direction.

t	0	1	2	3
x	10	10	30	10
y	10	30	10	10

eg 7	A toy car travels at a constant speed of 15 m/s CCW (counter clock wise) in a circle of radius 10m. Find the parametric equations of the path of the car.

Since $v = r\omega$, where $v = \frac{s}{t}$ the linear velocity, and $\omega = \frac{\theta}{t}$ the angular velocity
$v = r\omega \;\to\; v = r\frac{\theta}{t}$ $so\ \theta = \frac{3}{2}t.$ $x = 10\cos(\frac{3}{2}t)$ and $y = 10sin\,(\frac{3}{2}t.)$

Parametric Representation of Cartesian Curves
The parametric representation of Cartesian curves is not unique.

eg 8 Find a parametric representation for $y = x^2 - 2x + 1$ in [0,1].
$x = t, y = t^2 - 2t + 1$; $0 \le t \le 1$ is one representation, and $x = t + 1, y = t^2$; $-1 \le t \le 0$ is another representation.

eg 9 A spider on the floor is crawling such that its position at any time t seconds is given by C_1: $x = 2t+1, y = 2t$ for $t \ge 0$. At the same time, a small bug has a position C_2: $x = t+1, y = 2t^2$.

a)	Will the two paths intersect? At what point? ANS: Yes at (3/2, ½)
b)	Will the spider get the bug? At what point? ANS: Yes at (0, 0)

Check Graphically. (Hint: Change the parameter of C_2 to u. If the system of equations in t and u has a solution, the two paths intersect. If t and u are the same, the spider and the bug will meet since they will pass through the same point at the same time).

eg 10	The path of two objects in the x-y plane are given parametrically by $C_1 : x = t + 1, y = t^2$, and $C_2 :$ $x = t, y = 5 - 2t$.
Find algebraically the point of intersection of the two paths.

eg 11	The position of two objects in the x-y plane are given parametrically by $x = 3t, y = 2t^2$, and $x = 4t - 2, y = 4t$. Find algebraically if the two objects collide. If so, at what point? Ans: Yes at (6,8)

Homework

1. a) Graph $x = -\,cosh(t)$, $y = sinh(t)$, $-3 < t < 3$.
   b) Graph $x = sec(t)$, $y = tan(t)$, for $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $-\pi < t < \pi$.

2. Change to Cartesian by eliminating the parameter

1) $x = 2t + 4$, $y = t - 1$

2) $x = 1 - 2t$, $y = t^2 + 4$ $0 \le t \le 3$

3) $x = \sqrt{t}$, $y = t^3$

4) $x = 2\,cos(t)$, $y = \frac{1}{2}\,sin\,(t)$

5) $x = e^t$, $y = e^{-t}$

6) $x = 2 + \cos(t)$, $y = 3 + sin(t)$, $0 \le t \le 2\pi$

7) $x = 2\,cos(t)$, $y = sin^2(t)$

8) $x = 1/t$, $y = ln(t)$ $1 \le t \le e$

End Review

Equations of lines

To find the equation of a line, we need a point and a vector parallel to the line.
Let $\vec{r} = < x, y, z >$ be a position vector in space that describes a line.
If $\vec{r}_0 = < x_0, y_0, z_0 >$ is the position vector of a point (x_0, y_0, z_0) on a line, and $\vec{v} = < a, b, c >$ is a vector parallel to the line, the equation of the line can be written as $\vec{r} = \vec{r}_0 + t\,\vec{v}$ with t a parameter. This is called the vector form of the equation of a line.
Since $< x, y, z > = < x_0, y_0, z_0 > + t < a, b, c > = < x_0 + t\,a, y_0 + t\,b, z_0 + t_c >$, we can say
$x = x_0 + t\,a$; $y = y_0 + t\,b$; $z = z_0 + t\,c$. This is called the parametric form of the equation of a line.

Since $t = \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$, this is called the symmetric form of the equation of a line. The numbers a, b, c are called the direction numbers of the line.

eg 12 Find the equation of the line through the points (2, 4, –3) and (3, –1, 1).

Let $\vec{r}_0 = < 2, 4, -3 >$ be the position vector of a point on the line and $\vec{v} = < 3-2, -1-4, 1-(-3) >$ $= < 1, -5, 4 >$ be a vector parallel to the line.
The vector equation of the line will be given by $\vec{r} = < 2,4,-3 > + t < 1,-5,4 >$
The parametric form by $L : (x = 2 + t; y = 4 = 5t; z = -3 + 4t)$ and the symmetric form by
$\frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4}$.

The equation of the line can also be written with using the point (3,-1, 1) instead of using the point (2, 4, –3). So $\vec{r} = < 3, -1, 1 > + s < 1, -5, 4 >$ where the value of t and are different. If we equate the coordinates of both equations, we find that $t = s + 1$.

eg 13 Find the parametric equations for the line through the origin and parallel to the line segment $\vec{r}$ = < 0, – 1, 2 > +t < 1, 1, 0 > , $1 \le t \le 2$.
ans: $x = t, y = t, z = 0$

eg 14 Find the vector equation for the line containing the point (0, 2, 1) and perpendicular to the line $x = 2t, y = 1 - t, z = 2 + t$. Many answers are possible. ans: < 0, 2, 1 > + t < 0, 1, 1 >.

Line Segments

Consider the line segment from $\vec{r}_0$ = < x_0, y_0, z_0 > to $\vec{r}_1$ = < x_1, y_1, z_1 >. The equation of the segment will be given by $\vec{r} = \vec{r}_0 + t\,(\vec{r}_1 - \vec{r}_0)$ for $0 \le t \le 1$

eg 15 Find the line segment from the point (0, 2, – 1) to (2, 3, 1).
< 0, 2, – 1 > + t < 2, 1, 2 > = < $2t$, $2 + t$, $-1 + 2t$ > for $0 \le t \le 1$.

Skew Lines

Lines that are not parallel and do not intersect.
eg 16 The lines L_1 : ($x = 1 + t, y = -2 + 3t, z = 4 - t$) and are skew,
L_2 : ($x = 2t, y = 3 + t, z = -3 + 4t$)

Since the parallel vectors to the lines < 1, 3, –1 > and < 2, 1, 4 > are nor multiple of each other, the lines are not parallel.

Since the solution of $1 + t = 2s$ gives $s = \frac{4}{3}$ and $t = \frac{5}{3}$ when we solve the

$$-2 + 3t = 3 + s$$
$$4 - t = -3 + 4s$$

first and third equations. Since the lines do not intersect because these solutions do not satisfy the second equation, the two lines are skew.

Point of Intersection Between Two Lines

eg 17 Show that the lines $x = t - 1, y = t + 5, z = 1$ and $x = t - 3, y = -t + 1, z = t + 2$ intersect, and find the point of intersection.

If they intersect, we can find a value of t and s that satisfy the equations
$x = t - 1 = s - 3$, $y = t + 5 = -s + 1$, $z = 1 = s + 2$. Since the three sets of equations are satisfied by (s, t) = (– 1, – 3) the point of intersection (x, y, z) = (– 4, 2, 1)

Distance between a point and a line:

Review

The distance between a point and a line in 2 and 3 space will be given by $D = \frac{|\vec{u} \, x \, \vec{v}|}{|\vec{v}|}$ where $\vec{v}$ is a vector parallel to the line and $\vec{u}$ is the vector from a point in the line to the point off the line.

Proof:
Let P be any point on line, Q a point not on the line and $\vec{v}$ a vector parallel to the line. If $\vec{u}$ is the vector from P to Q and θ the angle between the two vectors, the distance between the point and the line will be given by $D = |\vec{u}|\sin(\theta) = \frac{\vec{v}}{\vec{v}}|\vec{u}|\sin(\theta) = \frac{|\vec{u}x\vec{v}|}{|\vec{v}|}$.

In 2D, if the equation of the line is given, the problem can be found easily using the formula
d = $\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$ where (x_1, y_1) is the point off the line.

eg 18 Find the distance between the point (- 3, 1,2) and the line through the points (1, 1, 0) and (–2,3, –4).

Since $\vec{u}$ = < 4, 0, - 2> and $\vec{v}$ = < – 3, 2, – 4 >, $D = \frac{|\vec{u}\,x\,\vec{v}|}{|\vec{v}|} =$

$$\frac{\left|\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 0 & -2 \\ -3 & 2 & -4 \end{vmatrix}\right|}{\sqrt{29}} = \frac{|< 4,22,8 >|}{\sqrt{29}} = \frac{2\sqrt{141}}{\sqrt{29}}.$$

End Review

eg 19 Find the distance between the point (– 2, 1,1) and the line $x = 3 - t$, $y = t$, $z = 1 + 2t$.

Since point on the line is P : (3, 0, 1) and a vector parallel to the line
$\vec{v}$ = < –1, 1, 2 > , $\vec{u}$ = < – 5, 1, 0 > and

$$D = \frac{|\vec{u}x\vec{v}|}{|\vec{v}|} = \frac{\left|\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -5 & 1 & 0 \\ -1 & 1 & 2 \end{vmatrix}\right|}{\sqrt{6}} = \frac{|<2,10,-4>|}{\sqrt{6}} = \frac{2\sqrt{30}}{\sqrt{6}} = 2\sqrt{5}.$$

Homework

1. Find the vector form for the equation of the line through the points
   P(- 3, 2, – 3) and Q (1, – 1, 4). ans: $\vec{L}$ = < -3, 2,-3 > + t < –4.3, –7 >.
2. Show that the lines $x = 2 + 8t$, $y = 6 - 8t$, $z = 10t$ and $x = 3 + 8t$, $y = 5 - 3t$, $z = 6 + t$ are skew.
3. Show that the lines $x = 4t - 1$, $y = t + 3$, $z = 1$ and $x = 12t - 13$, $y = 6t + 1$, $z = 3t + 2$ intersect, and find the point of intersections. [(–17, –1,1)].
4. Find the distance from the point P(1,1,5) to the line $\vec{L}$ = < 1, 3,0 > + t < 1, -1,2 > $[\sqrt{5}]$
5. Find the distance from the point P(2, 1, 3) to the line $x = 2 + 2t$, $y = 1 + 6t$, z = 3. [0; *point on line*]
6. Find the distance between the point (4, 3, 0) and the vector defined by the points (2, 1, 3) and (0, 2, -1). $[\frac{\sqrt{257}}{\sqrt{21}}]$
7. Verify that the lines $x = 2 - t$, $y = 2t$, $z = 1 + t$ and $x = 1 + 2t$, $y = 3 - 4t$, $z = 5 - 2t$ are parallel, and find the distance between them. $[\frac{\sqrt{35}}{\sqrt{6}}]$

Equations of Planes:

Review

The equation of a plane is determined by a point on the plane and a normal to the plane.
Let $\vec{r} = < x, y, z >$ be a position vector in space that describes a plane. If $\vec{r}_0 = < x_0, y_0, z_0 >$ is the position vector of the point (x_0, y_0, z_0) on the plane and $\vec{n} = < a, b, c >$ is the normal to the plane, the equation of the plane will be given by $(\vec{r} - \vec{r}_0).\ \vec{n} = 0$. So $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or
$ax + by + cz - (ax_0 + by_0 + cz_0) = 0$. We can write the equation as $ax + by + cz + d = 0$ where
$d = -(ax_0 + by_0 + cz_0)$.

eg 20 Find the equation of the plane through (2, 4, -1) with $\vec{n} = < 2, 3, 4 >$.
$2x + 3y + 4z - ((2)(2) + (3)(4) + (4)(-1)) = 0$, so $2x + 3y + 4z - 12 = 0$ is the equation of the plane.

eg 21 Give the equation of the plane through (1, – 1, 2) parallel to the plane $3x - 5y + 6z = 10$.

Since the normal of both planes are the same, $\vec{n} = < 3, -5, 6 >$, the equation of the parallel plane becomes $3x - 5y + 6z = 3 + 5 + 12$ or $3x - 5y + 6z = 20$

eg 22 Find the equation of the plane that contains the points (1, 3, 2), (3, – 1, 6), (5, 2, 0).

We can make the vectors $\vec{u} = < 2, -4, 4 >$ from the first two points and $\vec{v} = < 2, 3, -6 >$ from the last two. Since the normal to the plane is $\vec{n} = \vec{u} \times \vec{v} = < 12, 20, 14 > = < 6, 10, 7 >$, the plane is $6x + 10y + 7z - (6 + 30 + 14) = 0$ or $6x + 10y + 7z - 50 = 0$, where the point (1, 3, 2) was used as the point on the plane.

End Review

eg 23 Find the equation of the plane that contains the point (2, 0, 3) and the line $x = -1$, $y = t$, $z = 4 + 2t$.

Since the line contains the point (– 1, 0 4) and the vector < 0, 1, 2 >, we have two points and a vector on the plane. With the two points, we can build the vector < 3, 0, –1 > that when crossed with the other vector < 0, 1, 2 >, we obtain < 1, – 6, 3 > the normal of the plane. With the normal and a point, we obtain the equation of the line $x - 6y + 3z = 11$

eg. 24 Find the equation of the plane containing the line $x = 1 + 3t$, $y = 1 + 2t$, $z = 1$ and perpendicular to the plane $x + z = 5$. Ans: $2x-3y-2z = -3$

eg 25 Find the smallest angle between the plane $x + y + z = 1$ and $x - 2y + 3z = 1$.
The angle between two planes will be the angle between their normal vectors $\overrightarrow{n_1} = < 1, 1, 1 >$ and $\overrightarrow{n_2} = < 1, -2, 3 >$.

By the dot product, $cos(\theta) = \frac{\overrightarrow{n_1} \cdot \overrightarrow{n_2}}{|n_1||n_2|} = \frac{1-2+3}{\sqrt{3}\sqrt{14}} = \frac{2}{\sqrt{42}}$ so $\theta \approx 72^{o}$.

eg 26 Find the vector form of the line of intersection between the plane $x + y + z = 1$ and $x - 2y + 3z = 1$.

We need any point on the intersection of the planes and a vector parallel to the line of intersection. The point can be found by letting $z = 0$ so the system $x + y = 1$ will have solutions $x = 1$, $y = 0$. The point on
$x - 2y = 1$
on the intersection will be (1, 0, 0). The parallel vector will be the cross product of the two normals, so $\vec{v} = < 5, -2, -3 >$. The line will be given by $\vec{r} = < 1, 0, 0 > + t < 5, -2, -3 >$.

<u>eg 27</u> Find the point where the line L_2 : ($x = 2 + 3t$, $y = -4t$, $z = 5 + t$) intersects the plane $4x + 5y - 2z = 18$.

If we substitute x, y and z into the line, $4(2 + 3t) + 5(-4t) - 2(5 - t) = 18$, we find $t = -2$. At $t = -2$, $x = 2 + 3(-2)$, $y = -4(-2)$ and $z = 5 + (-2)$ so the point of intersection is $(-4, 8, 3)$.

If the dot product of the vector parallel to the line and the normal to the plane is zero, the line and the plane will not intersect. Why?

<u>eg 28</u> Find the parametric equation of the line through (5,2,0) that is parallel to the planes $x - y + z = 0$, and $-x - z + 1 = 0$.

Since the two planes are not parallel, the cross product of their normals will be parallel to the line.

Since $\overrightarrow{n_1}\, x\, \overrightarrow{n_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ -1 & 0 & -1 \end{vmatrix} = < 1, 0, -1 >$,

$\vec{r} = < 5, 2, 0 > + t \ < 1,0,-1 >$

Homework

1. Find the equation of the plane that contains both the point (1,2,3) and the line $x = 3t - 1$, $y = 6$, $z = t + 2$. [Ans: 4x-y-12z+34=0]

2. Find the equation of the plane that contains the point (2, 0, 3) and the line $x = -1 + t$, $y = t$, $z = -4 + 2t$. [Ans: –7x+y+3z+5=0]

3. Find the equation of the plane containing the line $x = -2 + 3t$, $y = 4 + 2t$, $z = 3 - t$ and perpendicular to the plane $x - 2y + z = 5$.
[Ans: 4y+8z-40=0].

4. Find the equation of the plane through (-1, 2, -5) and perpendicular to the planes $2x - y + z = 1$ and $x + y - 2z = 3$. [Ans: x+5y+3z+6=0]

5. Find the equation of the plane through (1,-1,0) and perpendicular to the planes $x - y + z = 0$, and $x + y - z + 1 = 0$. [Ans: x+z+1=0]

6. Find the angle between the planes $3x - 6y - 2x = 15$ and $2x + y - 2z = 5$. [Ans: $cos^{-1}\left(\frac{4}{21}\right)$]

7. Find the vector form for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect. [Ans: <3, –1,0> + t < 14,2,15 >]

8. Find the vector form of the line of intersection of the planes $-2x + 3y + 7z = -2$ and $x + 2y - 3z = -5$. [Ans: $\vec{r}$ = <–41,0, –12> + t < –23,1,-7>]

9. Find the vector form of the line of intersection of the planes $x - z = 2$ and $x - y = -5$. [Ans: $\vec{r} = < 0,5,-2 > + t < -1,-1,-1 >$]

10. Find the parametric equations of the line of intersection of the planes $-x + y + z = -1$ and $x + 2y + z = -2$. [Ans: $x = t\,, y = -2t - 1\,, z = 3t$]

11. Find the parametric equation of the line through (5,2,0) that is parallel to the planes $x - 4y + 2z = 0$, and $2x + 3y - z + 1 = 0$. [Ans: $x = 5-2t$, $y = 2+5t$, $z = 11t$]

Distance from a Point to a Plane:

Review

Let P_1 be the point (x_1, y_1, z_1) and $P_0 = (x_0, y_0, z_0)$ be a point on a plane. The distance from P_1 to the plane will be $D = |comp_{\vec{n}}\vec{u}| = \frac{|\vec{n}\cdot\vec{r}|}{|\vec{n}|}$ where $\vec{u} = < x_1 - x_0, y_1 - y_0, z_1 - z_0 >$ and $\vec{n}$ is a normal to the plane.

$D = \frac{|\vec{n}\cdot\vec{r}|}{|\vec{n}|} = \frac{|<a,b,c>\cdot<x_1-x_0, y_1-y_0, z_1-z_0>|}{\sqrt{a^2+b^2+c^2}} = \frac{|ax_1+by_1+cz_1-(ax_0+by_0+cz_0)|}{\sqrt{a^2+b^2+c^2}} = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$ where $d = -(ax_0 + by_0 + cz_0)$.

eg 29 Find the distance from the point $P_1 = (2, 1, 5)$ to the plane that contains the point $P_0 = (1, -1, 4)$ with normal $\vec{n}$ = < 2, 4, 1 >.

$D = \frac{|\vec{n}\cdot\vec{r}|}{|\vec{n}|} = \frac{|<2,4,1>\cdot<2-1,1+1,5-4>|}{\sqrt{2^2+4^2+1^2}} = \frac{|<2,4,1>\cdot<1,2,1>|}{\sqrt{2^2+4^2+1^2}} = \frac{11}{\sqrt{21}}$.

eg 30 Find the distance from the point $P_1(2, -3, 4)$ to the plane $x + 2y + 2z = 13$.

$D = \frac{|ax_1-by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} = \frac{|1(2)+2(-3)+2(4)-13|}{\sqrt{1^2+2^2+2^2}} = \frac{|-9|}{3} = 3$

Homework

1. Find the distance from the point S(1,1,3) to the plane $3x + 2y + 6z = 6$. $\left[\frac{17}{7}\right]$
2. Find the distance from the point (2, - 3, 4) to the plane $x + 2y + 2z = 6$. $\left[\frac{2}{3}\right]$

End Review

eg 31 Find the distance between the planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

Since the two planes are parallel (their normals are multiple or each other), we need a point on each plane and the normal to find the distance.
A point on the first plane is $P_1 : (\frac{1}{2}, 0,0)$ and a point on the other plane $P_0 = (0, 1, 0)$.

Since $\vec{n}$ = < 5, 1, – 1 > and $\vec{r}$ = < $\frac{1}{2}$, - 1, 0 > , $D = \frac{|\vec{n}\cdot\vec{r}|}{\vec{n}} = \frac{\frac{3}{2}}{\sqrt{27}} = \frac{1}{2\sqrt{3}}$.

Another way to solve this problem is by using one plane and a point on the other plane with the formula of the distance from a point to a plane $D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$. From the plane $5x + y - z = 1$ we obtain a = 5, b = 1, c = –1 and d = – 1 so by using the point $(\frac{1}{2}, 0,0)$ on the first plane,
$D = \frac{\left|5\left(\frac{1}{2}\right) + 1(0)+1(0)-1\right|}{\sqrt{27}} = \frac{1}{2\sqrt{3}}$.

eg 32 Use the formula for the distance between a point and a plane to show that the distance between the parallel planes $ax + by + cz = d_1$ and $ax + by + cz = d_2$ is given by $D = \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

eg 33 Use the $D = \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$ to find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.
If we divide the first plane by 2, d_1=5/2. Since d_2=1, $D = \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}} = \frac{|5/2-1|}{\sqrt{5^2+1^2+(-1)^2}} = \frac{1}{2\sqrt{3}}$.

eg 34 Find the distance between the skew lines $L_1 : (x = 1 + t, y = 2 + t, z = 1 + t)$.
$L_2 : (x = 1 + t, y = 2 - t, z = 2 + t)$

To find the distance, since the two lines lie on parallel planes, we need the equation of one of the planes and a point on the line of the other plane.

If we cross the vectors parallel to the lines we obtain the normal to the planes.
So $\vec{n}$ = < 1, 1, 1 > x < 1, – 1, 1 > = < 2, 0, – 2 >.

Since (1, 2, 2) is a point on L_2, $P_2 : 2x + 0y - 2z - (2(1) + 0(2) - 2(2)) = 0$ or $2x - 2z + 2 = 0$.

Since (1, 2, 1) is a point of L_1, $D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} = \frac{|2(1)+0(2)-2(1)+2|}{\sqrt{2^2+0^2+(-2)^2}} = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$.

Homework

1. Find the distance between the planes $10x + 2y - 2z = 4$ and $5x + y - z = 1$. Ans: $\frac{1}{\sqrt{27}}$
2. Find the distance between the planes $6x + 2y - 2z = 4$ and $3x + y - z = 4$. Ans: $\frac{2}{\sqrt{11}}$
3. Find the distance between the planes $10x + 2y - 2z = 8$ and $5x + y - z = 1$. Ans: $\frac{3}{\sqrt{27}}$

Homework 17.1

1. Which of the following statements are true and which are false for all lines and planes in *xyz*-space: (a) Two lines parallel to a third line are parallel. (b) Two lines perpendicular to a third line are parallel. (c) Two planes parallel to a third plane are parallel. (d) Two planes perpendicular to a third plane are parallel. (e) Two lines parallel to a plane are parallel. (f) Two lines perpendicular to a

plane are parallel. (g) Two planes parallel to a line are parallel. (h) Two planes perpendicular to a line are parallel. (i) Two lines are either parallel or intersect. (j) Two planes either are parallel or intersect. (k) A line and a plane either are parallel or intersect. (l) Two non-parallel lines either intersect or there are parallel planes that contain them.
Ans: (a) T, (b) F, (c) T, (d) F, (e) F, (f) T, (g) F, (h) T, (i) F, (j) T, (k) T, (l) T

2. Give equations of (a) the line through (5, 6, 7) and parallel to the line $x = 4$, $y = 6 - t$, $z = 9 + 2t$, (b) the line through the origin and perpendicular to the plane $3x - 4y + 5z - 18 = 0$
Ans: a) $\langle 5, 6-t, 7+2t \rangle$ b) $\langle 3t, -4t, 5t \rangle$

3. Find the plane through (10, 5, 0) and perpendicular to the line $x = 3 + t$, $y = 4 - 2t$, $z = 3t$.
Ans: $x - 2y + 3z = 0$

4. Give (a) parametric equations of the line through (6, 0, –3) and parallel to the planes $2x - 4y = 7$ and $3y + 5z = 0$, (b) an equation for the plane through (0, 1, 0) and parallel to **i** + **j** and to **j** – **k**, (c) an equation for the plane through (5, -1, -2) and perpendicular to the planes $y - z = 4$ and $x + z = 3$, (d) an equation for the plane through (2, 0, 0) and perpendicular to the planes $z = 4$ and $x + y + z = 0$, (e) and equation for the plane through (2, 2, 4), (5, 6, 4), and (1, 3, 5), (f) an equation for the plane through (1, 2, 3) and parallel to the plane $4x - y + 3z = 0$ (g) an equation for the plane through the origin and parallel to the plane $3x - y + z = 1000$.
Ans: a) $\langle 6-10t, -5t, -3+3t \rangle$ b) $-x + (y-1) + z = 0$ c) $(x-5) - (y+1) - (z+2) = 0$
d) $-(x-2) + y = 0$ e) $4(x-2) - 3(y-2) + 7(z-4) = 0$ f) $4x - y + 3z = 11$ g) $3x - y + z = 0$.

5. How are the numbers a and b related if $< a, b, 5 >$ is parallel to the plane $4x - 2y + z = 5$?
Ans: $4a - 2b + 5 = 0$

6. Find an equation of each of the following planes and then show how it relates to the coordinates axes by drawing a triangular or rectangular portion of it that includes its intercepts: (a) the plane with x-intercept 2, y-intercept 3, and z-intercept 5, (b) the plane parallel to the x-axis with y-intercept 5 and z-intercept 2, (c) the plane parallel to the yz-plane with x-intercept 10.
Ans: $y/2+y/3+z/5 = 1$, b) $y/5+z/2 = 1$, c) $x=10$.

7. Find the point where the lines L_1: $x = 2t$, $y = 3 - t$, $z = -2 + 4t$ and L_2: $x = 4 + t$, $y = 1 + 2t$, $z = 6$ intersect. (Replace t by s in the second set of equations and solve for t and s.) Ans:(4,1,6)

8. Find the distance between the planes $2x + 2y + z = 5$ and $2x + 2y + z = 25$. Ans: 20/3

9. Give parametric equations of the line that is perpendicular to the lines L_1: $x = 2+t$, $y = 4 - t$, $z = 6 + 2t$ and L_2: $x = 2 - t$, $y = 3 + 2t$, $z = 7 - 3t$ and passes through their point of intersection.
Ans: $\langle 1-t, 5+t, 4+t \rangle$

10. (a) Show that the line L: $x = 2t - 3$, $y = 4t - 2$, $z = 6$ is parallel to the plane $2x - y + z = 0$.
(b) Give an equation of the plane parallel to $2x - y + z = 0$ that contains the line L.
Ans: b) $2(x+3) - (y+2) + (z-6) = 0$

17.2 Motion, Velocity and Acceleration

If $\vec{r}\,(t)$ gives the path (position, trajectory) of an object at any time along a curve at any value of t, $\frac{d\vec{r}\,(t)}{dt} = \vec{v}(t)$ is the velocity of the object along that curve in the direction of the object. The speed of the object is given by $|\vec{v}(t)|$. The length (magnitude) of the vector will represent the speed of the object. The acceleration will be given by $\frac{d\vec{v}(t)}{dt} = \vec{a}(t)$

<u>eg 34</u> Given the position of an object parametrically as $x = 4cos(t)$, $y = 3sin\,(t)$, $0 \le t \le 2\pi$ at any value of t, find the velocity, the speed and the acceleration at $t = \frac{3\pi}{4}$.

Since the position vector is given by $\vec{r}(t) = < 4cos(t), 3sin(t) >$, $\vec{v}(t) = \frac{d\vec{r}\,(\frac{3\pi}{4})}{dt} = < -4sin(t), 3cos(t) > \Big|_{t=\frac{3\pi}{4}} = < -\frac{4}{\sqrt{2}}, -\frac{3}{\sqrt{2}} >$, the speed is $|\vec{v}(t)| = \frac{5}{\sqrt{2}}$ and $\vec{a}(t) = \frac{d\vec{v}\left(\frac{3\pi}{4}\right)}{dt} = < -4cos(t), -3sin(t) > \Big|_{t=\frac{3\pi}{4}} =$
$< \frac{4}{\sqrt{2}}, -\frac{3}{\sqrt{2}} >$

<u>eg 35</u> If $\vec{v}(t)$ is the velocity and $\vec{r}(t)$ the position of a particle, show that when the speed $|\vec{r}\,(t)| = C$ a constant, then $\vec{v}\,(t)$ and $\vec{r}(t)$ are perpendicular.

$|\vec{r}(t)|^2 = \vec{r}\,(t) \cdot \vec{r}\,(t) = C^2$. If we differentiate $\vec{r}\,(t) \cdot \vec{r}\,(t) = C^2$ we obtain

$\vec{v}(t) \cdot \vec{r}\,(t) + \vec{r}\,(t) \cdot \vec{v}(t) = 2\vec{r}(t) \cdot \vec{v}(t) = 0 \Rightarrow \vec{r}(t)$ is perpendicular to $\vec{v}(t)$.

We know that in a circle $|\vec{r}\,(t)|$ is constant and $\vec{r}(t)$ is perpendicular to $\overrightarrow{r}(t) \doteq \vec{v}(t)$.

<u>eg 36</u> If $\vec{r}(t)$ the position and $\vec{v}(t)$ the velocity of a particle, show that $\frac{d}{dt}|\vec{r}(t)| = \frac{\vec{v}\,(t)\cdot\vec{r}\,(t)}{|\vec{r}\,(t)|}$

$\frac{d}{dt}\,|\vec{r}\,(t)|^2 = \frac{d}{dt}\vec{r}(t) \cdot \vec{r}(t) = 2\,\vec{r}(t) \cdot \vec{v}(t)$, but also $\frac{d}{dt}\,|\vec{r}\,(t)|^2 = 2|\vec{r}\,(t)|\,\left(\frac{d}{dt}|\vec{r}\,(t)|\right)$.

If we equate both equations $\frac{d}{dt}|\vec{r}\,(t)| = \frac{\vec{v}\,(t)\cdot\vec{r}(t)}{|\vec{r}(t)|}$.

<u>eg 37</u> If $\vec{v}(t)$ is the velocity and $\vec{r}\,(t)$ the position of a particle, find $\frac{d}{dt}\,(\vec{r} \times \vec{v})$.

$\frac{d}{dt}(\vec{r} \times \vec{v}) = \left(\frac{d}{dt}\,\vec{r}\right) \times \vec{v} + \vec{r} \times \frac{d}{dt}\,\vec{v} = \vec{r} \times \vec{a}$ since $\vec{v} \times \vec{v} = 0$ and $\frac{d}{dt}\,\vec{v} = \vec{a}$ the acceleration.

Integration with Initial Conditions

<u>eg 38</u> If $\vec{v}(t) = < cos\,t, sin\,t, t >$ is the velocity of a particle, find the position $\vec{r}(t)$ if the position $\vec{r}\,(\pi/2) = \hat{\imath} = < 0, 1, 0 >$

Use $\vec{r}(t) = \int \vec{v}(t)dt = \int < cos\,t, sin\,t, t > dt = < -sin(t), cos(t), \frac{t^2}{2} > + \overrightarrow{C}$ $\overrightarrow{r}\,(\pi/2) = < 1, 0, \frac{\pi^2}{8} > + \overrightarrow{C} = < 0, 1, 0 > \Rightarrow + \overrightarrow{C} = < -1, 1, \frac{\pi^2}{8} >$ so $\overrightarrow{r}(t) = < sin(t) - 1, cos(t) + 1, \frac{t^2}{2} - \frac{\pi^2}{8} >$.

Vector Form of Newton's Law of Motion

To study motion on curves we need Newton's law in vector form $\vec{F}(t) = m\,\vec{a}(t)$.

eg 39 If $\vec{F}(t) = < 2e^t, 2e^{-t}, 2\sqrt{2}\,t >$ is the force on a particle of a 2 slug mass moving in *xyz* space with the distances measured in feet, find the position $\vec{r}(t)$ if the velocity $\vec{v}(0) = \hat{\imath}$, and the position $\vec{r}(0) = \hat{\jmath}$.

Since $\vec{F}(t) = m\,\vec{a}(t) = 2\vec{a}(t)$, $\vec{a}(t) = < e^t, e^{-t}, \sqrt{2}\,t >$, $\vec{v}(t) = \int \vec{a}(t)dt = < e^t, -e^{-t}, \frac{\sqrt{2}t^2}{2} > + \vec{C_1}$.

If we apply the IC $\vec{v}(0) = \hat{\imath}$, $\vec{v}(0) = < 1, -1, 0 > + \vec{C}_1 = < 1, 0, 0 > \Rightarrow \vec{C}_1 = < 0, 1, 0 >$ so $\vec{v}(t) = < e^t, -e^{-t} + 1, \frac{\sqrt{2}t^2}{2} >$.

Since $\vec{r}(t) = \int \vec{v}(t)dt = < e^t, -e^{-t} + t, \frac{\sqrt{2}t^3}{6} > + \vec{C}_2$, If we apply the IC $\vec{r}(0) = \hat{\jmath}$, $\vec{r}(0) = < 1, 1, 0 > + \vec{C}_2 = < 0, 1, 0 > \Rightarrow \vec{C}_2 = < -1, 0, 0 >$ so $\vec{r}(t) = < e^t - 1, e^{-t} + t, \frac{\sqrt{2}t^3}{6} >$.

Length of a Curve

If $y = f(x)$, the length of a curve in $[a, b]$ is given by $L = \int_a^b \sqrt{1 + (f(x)')^2}dx$.

If the curve is given by $\vec{r}(t) = < x(t), y(t) >$ or $\vec{r}(t) = < x(t), y(t), z(t) >$, the length in $a \le t \le b$ is given by $L = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ dt or $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

eg 40 Find analytically the arc length of $x = \frac{t^2}{2}$, $y = \frac{t^3}{3}$ in the interval $0 \le t \le 1$.

$L = \int_a^b \sqrt{(t)^2 + (t^2)^2}\; dt = (2\sqrt{2} - 1)/3$

Arc Length as a Parameter

For many purposes we want to express a curve as a parameter of arc length. *A* curve parameterized in terms of the arc length *s*, is independent of the coordinate system. The arc length as a function of *t* from the point t_0 to any point *t* will be defined by $s(t) = \int_{t_0}^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du.$

Since $\vec{r} = < x, y, z >$, $s(t) = \int_{t_0}^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2 du}$ can be written as $s(t) = \int_{t_0}^t \left|\frac{d\vec{r}\,(u)}{du}\right| du.$

eg 41 Consider $\vec{r}(t) = \; = < 3cos(2t), 3sin(2t) >$, $0 \le t \le 2\pi$

s(t) = $\int_0^t \sqrt{(-6\sin(2u))^2 + (6\cos(2u))^2}\, du = 6t$

so $\vec{r}(s) =< 3\cos\left(\frac{s}{3}\right), 3\sin\left(\frac{s}{3}\right) >, 0 \le s \le 12\pi$ is the parameterization of the curve in terms of *s*.
$\vec{r}(s)$ means that the length of the curve from the point $\vec{r}(0)$ to the point $\vec{r}(s)$ is *s* units long.
$\vec{r}(\pi) = < \frac{3}{2}, \frac{3\sqrt{3}}{2} >$, means that the length of the curve from the point (3,0) to the point $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ is π units long.

$$L = \int_0^\pi \sqrt{\left(-\sin\left(\frac{s}{3}\right)\right)^2 + \left(\cos\left(\frac{s}{3}\right)\right)^2}\, ds = \pi.$$

If we take the derivative with respect to *t* of $s(t) = \int_{t_0}^t \left|\frac{d\vec{r}(u)}{du}\right| du$, by part 1 of the Fundamental Theorem of Calculus we obtain $\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^t \left|\frac{d\vec{r}(u)}{du}\right| du = |\dot{\vec{r}}(t)|$.

This says that the change in arc length with time equals the magnitude of the tangent vector.

Note:
The Fundamental Theorem of Calculus for Integral of Derivatives
Suppose *F* (*x*) continuous in (*a*, *b*) and *F'* (*x*) exists for all but a finite numbers of points,
$\int_a^b F'(x)dx = f(b) - f(a)$ where $\frac{d}{dx} f(x) = F'(x)$.

The Fundamental Theorem of Calculus for Derivative of Integrals (Leibniz Rule)
Suppose $F(x) = \int_{g(x)}^{h(x)} f(t)dt$ is defined for all *x* in an interval *I*,
$\frac{d}{dx} F(x) = \frac{d}{dx} \int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x) - f(g(x))g'(x)$ where *f* (*h*(*x*)) and *f* (*g*(*x*)) are continuous in *I*.

Homework 17.2

1. If $\vec{a}(t) =< t^3, \frac{1}{2}\sqrt{6}t^2, t >$ is the acceleration of a particle, find the position $\vec{r}(t)$ is the velocity $\vec{v}(0) = \hat{k}$, and the position $\vec{r}(0) = \hat{\imath} + \hat{\jmath}$
   Ans: r= < t^5/20 +1, sqrt6 t^4/24+1, t^3/6 +t >.
2. A 2 slug object is moving in *xyz* space with the distances measured in feet.
   At *t* = 0, the object is at the point (3,0,2), and at *t* = 2 the velocity vector is < 5, 0, 5 >.
   Give a formula for the force vector if there is no air resistance.
   Ans: <5t+3,0,-16t^2+69t+2>
3. Find the arc length for $\vec{r}(t) = < \frac{t^3}{3}, \frac{\sqrt{2}\, t^2}{2}, t >$ for 0 < *t* < 1. Ans: 4/3
4. Find the arc length for $\vec{r}(t) = < t^2, 2t, \ln(t) >$ for 1 < *t* < *e*. Ans: e^2
5. Parameterize the length $\vec{r}(t) =< 3sin(t), 4t, 3cos(t) >, 0 \le t \le 2\pi$. Ans: s=5t
   $\vec{r}(s) =< 3sin\left(\frac{s}{5}\right), \frac{4s}{5}, 3cos\left(\frac{s}{5}\right) >, 0 \le s \le 10\pi$

17.3 Vector Fields

A vector field is a function that assigns to each point in R_2 or R_3 a vector.
In R_2, $\overrightarrow{F}(x, y) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath} = < P(x, y) , Q(x, y) >$.

eg 42 Sketch $\overrightarrow{F}(x, y) = < x, y >$ and $\overrightarrow{G}(x, y) = < -y, x >$. Use the Vector Field Analyzer found at
http://math.la.asu.edu/~kawski/vfa2/vfa2sample.html

In R_3, $\vec{F}(x, y, z) = P(x, y, z)\hat{\imath} + Q(x, y, z)\hat{\jmath} + R(x, y, z)\hat{k}$
$= < P(x, y, z) , Q(x, y, z) , R(x, y, z) >$.

eg 43 Sketch $\overrightarrow{F}(x, y, z) = < 2x, 2y, 2z >$ Use the 3D Vector Field Plotter found at
http://cose.math.bas.bg/webMathematica/MSP/Sci_Visualization/3DVectorField

Gradients Fields
Important classes of vector fields are the gradient fields.
A gradient vector field is given by $\vec{\nabla} f\ (x, y) = < f_x(x, y), f_y\ (x, y) >$ in $\Re^2$ or
$\vec{\nabla} f\ (x, y, z) = < f_x\ (x, y, z), f_y\ (x, y, z), f_z(x, y, z) >$ in $\Re^3$.
If a vector field $\vec{F}$ can be represented as the gradient of some scalar function f, ($\vec{F} = \vec{\nabla} f$), the vector field $\vec{F}$ is called conservative, and f is called the potential function of $\overrightarrow{F}$.

The Gravitational Force Field is an example of a Gradient Field.

eg 44 Show that $f(x, y, z) = \frac{GMm}{\sqrt{x^2+y^2+z^2}}$ is the potential function of the gravitational force field $\overrightarrow{F}$.

Newton's Law of Gravitation is given by $|\vec{F}| = \frac{GMm}{|\vec{r}|^2}$ in the direction of
$\vec{r} = < x, y, z >$ towards the origin. So $\vec{F} = \frac{-GMm}{|\vec{r}|^2}\hat{r} = \frac{-GMm}{|\vec{r}|^2}\frac{\vec{r}}{|\vec{r}|} = \frac{-GMm}{|\vec{r}|^3}\vec{r}$.

Since $\vec{\nabla}\left(\frac{GMm}{\sqrt{x^2+y^2+z^2}}\right) = \frac{<-GMmx, -GMmy, -GMmz>}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{-GMm}{|\vec{r}|^3}\vec{r}$, $f(x, y, z) = \frac{GMm}{\sqrt{x^2+y^2+z^2}}$ is the potential function of the gravitational field. In other words, the gravitational field is a conservative field with potential function $f(x, y, z) = \frac{GMm}{\sqrt{x^2+y^2+z^2}}$

Coulomb's Law:
The force between two electrically charged particles at a distance $\vec{r}$ is given by $\vec{F} = \frac{Qq}{4\pi\epsilon|\vec{r}|^3}\vec{r}$.
If $Qq > 0$ (like charges) the particles repel, if $Q\,q < 0$ (unlike charges) the particles attract.

The electric field of q will be given by $\vec{E} = \frac{\vec{F}}{q} = \frac{Q}{4\pi\epsilon|\vec{r}|^3}\,\vec{r}$.

The electric field is another example of a conservative potential field, so $\vec{E} = \vec{\nabla}\,\phi$ where $\phi = -\frac{Q}{4\pi\epsilon\sqrt{x^2+y^2+z^2}} = -\,\frac{Q}{4\pi\epsilon|\vec{r}|}$ is the potential.

HOMEWORK

1. Sketch the vector fields (without using technology) (a) <1,y> (b) <y,y> (c) <-x,-y> (d) <y,1>.
2. Show that $\vec{E} = \frac{Q}{4\pi\epsilon|\vec{r}|^3}\,\vec{r}$ is a conservative electric field with potential $\phi = -\,\frac{Q}{4\pi\epsilon|\vec{r}|}$.

17.4 The flow of a Vector Field

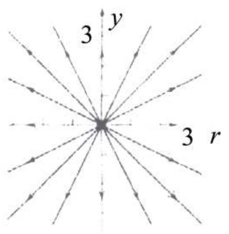
Velocity Field of a Fluid
Consider the velocity field $\vec{v}(t) = <\dot{x}\,(t), \dot{y}(t)>$ of a thin layer of water.
The streamlines of the field are the path C : $x = x(t)$, $y = y(t)$ or $\vec{r}(t) = < x(t), y(t) >$ taken by the particles of the fluid.

If the field is constant, all the particles will have the same velocity at all times.

eg 44 If $\vec{v}(t) = <\frac{3}{4}, 4>$ is a constant, the streamlines of the field will be $\vec{r}(t) = <\frac{3}{4}t + x_0,\ 4t + y_0>$ where we have integrated $\vec{v}(t)$ and applied the IC. $x(0) = x_0$, $y(0) = y_0$.

If we eliminate the parameter t, the path of a particle is $y = \frac{16}{3}(x - x_0) + y_0$ a line with slope $\frac{16}{3}$.

eg 45 If $\vec{v}(t) = <2x(t), 2y(t)>$ is not a constant field, since $\frac{dx}{dt} = 2x$ and $\frac{dy}{dt} = 2y$, the path the particles will be given by $\vec{r}(t) = < x_0e^{2t}, y_0e^{2t} >$ with IC $x(0) = x_0$, $y(0) = y_0$. If we eliminate the parameter, the path is $y = \frac{y_0}{x_0}x$ a the line through the origin with slope $\frac{y_0}{x_0}$.



eg 46 If $\vec{v}(t) = <-rsin(t), rcos(t)> = <-y, x>$ a rotational velocity field, the path the particles will be given by $\vec{r}$(t) = < $rcos(t)$, $rsin(t)$ > with IC $x(0) = y(0) = 0$. These are parametric equations of circles of radius r in a ccw. direction.