Integrating Functions of Several Variables

16.1 The Definite Integral of a Function of Two Variables

Consider $z = f(x, y)$ continuous on a bounded region R on the x-y plane. If we divide R into n sub-regions $R_1 \ldots R_n$ of areas $\Delta A_1 \ldots \Delta A_n$ respectively, with each sub region R_k containing the point $P_k(x_k, y_k)$,

$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \int_R f(x,y)dA$, where dA can be $dxdy$ or $dydx$ in the Cartesian coordinate system.

$\int_a^b f(x)\, dx$ is the area under the curve $y = f(x)$ in $[a, b]$, and if $f(x) = 1$, $\int_a^b f(x)\, dx$ is the length $[a, b]$. If we extend this concept one dimension, we can say $\int_a^b \int_c^d f(x,y)d\,A$ is the volume under the surface $z = f(x, y)$ in the rectangle $[a, b] \times [c, d]$, and if $z = f(x, y) = 1$, $\int_a^b \int_c^d f(x,y)d\,A$ is the area of the rectangle.

Average Value of a Function

If $f(x, y)$ is piecewise continuous in a bounded region R with piecewise smooth boundary, then $f_{ave} = \frac{1}{area}\int_R f(x,y)dA$.

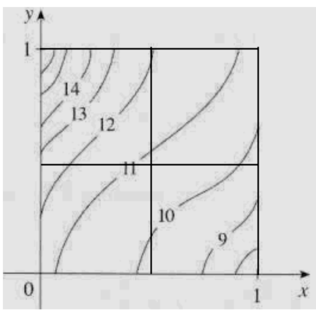
eg 1 The table below gives the value of $z = f(x, y)$. R is the rectangle $1 \le x \le 1.2$, $2 \le y \le 2.4$. Find the Riemann sums which are a reasonable under and over-estimates for $\int_R f(x,y)dA$. with $\Delta x = 0.1$ and $\Delta y = 0.2$. Find the average value of f in R.

y/x	1.0	1.1	1.2
2.0	5	7	10
2.2	4	6	8
2.4	3	5	4

For $\Delta x = 0.1$ $\Delta y = 0.2$, the over-estimate will be the sum of the area of each square times the largest value the function takes at any of the corners any sub-rectangle. Since $\Delta x = 0.1 \; x \; \Delta y = 0.2$, the area of each square will be $\Delta x \, \Delta y = 0.02$. The upper sum will be 0.02(7+6+10+8) = 0.62. The lower sum will be 0.02(4+3+6+4) = 0.34
For $\Delta x = 0.1$ $\Delta y = 0.2$, we can say that $0.34 < \int_R f(x,y)dA < 0.62$.

We can get a better estimate if we average the two sums, or $\int_R f(x,y)dA \approx \frac{0.34+0.62}{2} = 0.48$.
The average value of the function will be $f_{ave} = \frac{1}{total\ area}\int_R f(x,y)dA = \frac{0.48}{(0.2)(0.4)} = 6$.

eg 2 The figure below shows the contours of $f(x, y)$ on a square R. Using $\Delta x = \Delta y = \frac{1}{2}$, find Riemann sums which are reasonable over and under-estimate for $\int_R f(x,y)dA$. Repeat the same problem using $\Delta x = \Delta y = 1$



For $\Delta x = \Delta y = \frac{1}{2}$, the over-estimate will be the sum of the area of each square times the largest value the function takes at any of the corners of that sub-rectangle. If we start counterclockwise with the square at (0, 0), an over-estimate of the integral will be $\frac{1}{4}(12.8 + 10.8 + 12 + 17) = 13.15$.

For $\Delta x = \Delta y = \frac{1}{2}$, the under-estimate will be the sum of the area of each square times the smallest value the function takes at any of the corners of that sub-rectangle. If we start counterclockwise with the square at (0, 0), an under-estimate of the integral will be $\frac{1}{4}(9.8 + 7 + 9.8 + 10.8) = 9.35$.

For $\Delta x = \Delta y = \frac{1}{2}$, we can say that $9.35 < \int_R f(x,y)dA < 13.15$.

We can get a better estimate if we average the two sums, or $\int_R f(x,y)dA \approx \frac{13.15+9.35}{2} = 11.25$.

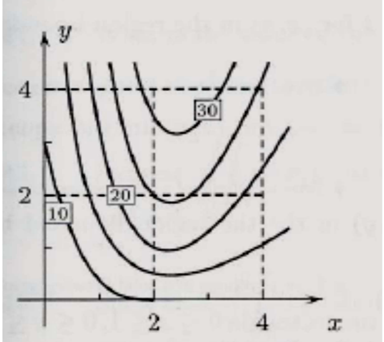
If we used the value of the function at the center if the squares, we obtain $\frac{1}{4}(13 + 11 + 9.8 + 10.8) = 11.15$.

For $\Delta x = \Delta y = 1$, the over-estimate will be 17 and an under-estimate 7. If we average the two values, we get $\int_R f(x,y)dA \approx \frac{17+7}{2} = 12$.

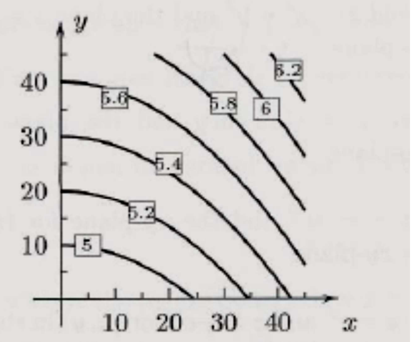
Homework 16.1

1. Approximate the Riemann sum for $\int_R M(x,y)dA$ in $0 \le x \le 4$, $0 \le y \le 4$ using four partitions as shown in the figure below. Ans: 312

2. Approximate the Riemann sum for $\int_R N(x,y)dA$ in $0 \le x \le 40$, $0 \le y \le 40$ using one partitions as shown in the figure below. Ans: 8480



Level curves of $M(x,y)$



Level curves of $N(x,y)$

3. Table below gives the values of the function $P(x,y)$. Find an over and underestimate of $\int_R P(x,y)dA$ to approximate the integral in $0 \le x \le 6$, $0 \le y \le 2$ by using four partitions

y/x	0	3	6
0	3	4	6
1	4	5	7
2	5	7	10

Ans: Over 87; under 48; estimate 67.5

16.2 Iterated Integrals

Iterated Integrals over Rectangular Regions

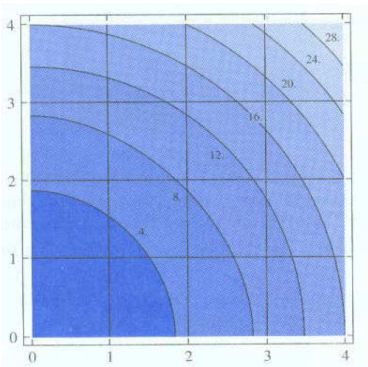
If I is an iterated integrals over a rectangular region, the integration can be switched.

Fubini's Theorem:
If $f(x, y)$ is continuous in the rectangle $R = \{(x, y) \mid a \le x \le b,\ c \le y \le d\}$, then

$$I = \int_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

eg 3 $\int_0^1 \int_0^2 (x + y^2)dydx = \int_0^1 (xy + \frac{y^3}{3})|_{y=0}^{y=2} dx = \int_0^1 (2x + \frac{8}{3})dx = \frac{11}{3}$.
$\int_0^2 \int_0^1 (x + y^2)dxdy = \int_0^2 (\frac{x^2}{2} + y^2 x)|_{x=0}^{x=1} dy = \int_0^2 (\frac{1}{2} + y^2)dy = \frac{11}{3}$.

eg 4 The figure below shows the contours of $f(x, y) = x^2 + y^2$. Using $\Delta x = \Delta y = 1$, estimate $\int_R f(x,y)dA$by finding the average of an over-estimate and under-estimate of the integral for $0 \le x, y \le 3$. Repeat the problem by evaluating the integral.



Exact will be $\int_0^3 \int_0^3 (x^2 + y^2)\, dxdy = \int_0^3 9 + 3y^2\, dy = 54.$
The over-estimate will be 1(3 + 6 + 10 + 13 + 9 + 6 + 10 +13 + 18) = 91.
The under-estimate will be 1(0 + 2 + 5 + 6 + 3 + 2 + 5 + 6 +9) = 30.
The average will be $\int_R f(x,y)dA \approx 60.5$ with relative error in 12%

eg 5 $\int_1^2 \int_0^\pi \big(xcos(xy)\big)dxdy$ has to be integrated by parts.
Let $u = x$, $du = dx$; $dv = cos(xy)$; $v = sin(xy)/y$.
So $\int_1^2 \int_0^\pi \big(x\cos(xy)\big)dxdy = \int_1^2 (\frac{x}{y}\sin(xy))\,|_0^\pi - \int_0^\pi (\sin(xy)/y)dx)dy =$
$\int_1^2 (\frac{\pi}{y}\sin(\pi y) + \frac{1}{y^2}\cos(\pi y) - \frac{1}{y^2})dy = \int_1^2 \frac{\pi}{y}\sin(\pi y)\, dy + \int_1^2 \frac{1}{y^2}\cos(\pi y)\, dy - \int_1^2 \frac{1}{y^2} dy.$
If we integrate by parts the first integral with $u = \frac{1}{y}$; $du = -\frac{1}{y^2}$; $dv = \sin(\pi y)\, dy$; and
$v = -\cos(\pi y)/\pi$, the integral becomes
$-\frac{1}{y}\cos(\pi y)\,|_1^2 - \int_1^2 \frac{1}{y^2}\cos(\pi y)\, dy + \int_1^2 \frac{1}{y^2}\cos(\pi y)\, dy + \frac{1}{y}|_1^2 = -2.$
This integral will be easier to compute if we switch the integrals.

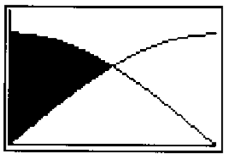
$\int_0^\pi \int_1^2 \left(x\cos(xy)\right)dydx = \int_0^\pi \sin(xy)|_{y=1}^{y=2}\, dx = \int_0^\pi (\sin(2x) - \sin(x))dx = -2$

<u>Iterated Integrals over General Regions</u>

In general, the limits of integration do not need to be over rectangular regions. Since an integral is a function of its limits, the inside integral should be a function of the outside variable of integration.

<u>Type I</u>
If the region of integration in the *x*-*y* plan is bounded above by $y(x) = g_2(x)$ and below by $y(x) = g_1(x)$ in $a < x < b$, the double integral will be $\int_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f\,(x,y)dydx$

<u>eg 6</u> Find the area between $y = \sin(x)$ and $y = \cos(x)$ in $0 \le x \le \pi/4$.

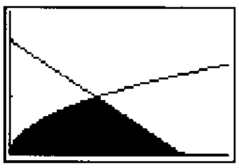


$\int_0^{\pi/4} \int_{\sin(x)}^{\cos(x)} dydx = \int_0^{\pi/4} (\cos(x) - \sin(x))dx = \sin(x) + \cos(x)|_0^{\pi/4} = \sqrt{2} - 1.$

<u>eg 7</u> Evaluate $\int\int_R e^{x^2} dA$ where *R* is the triangle (0, 0), (1, 0), (1, 1). Since the region is bounded above by $y(x) = x$ and below by $y = 0$ in $0 < x < 1$. The integral becomes $\int_0^1 \int_0^x e^{x^2}\, dydx = \int_0^1 xe^{x^2} dx = \frac{e^{x^2}}{2}|_0^1 = \frac{(e-1)}{2}$.

<u>Type II</u>
If the region of integration in the *x*-*y* plane is bounded at the right by $x(y) = h_2(y)$ and at the left by $x(y) = h_1(y)$ in $c < y < d$, the double integral will be $\int_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$

<u>eg 8</u> Consider $\int_R ydA$ where R area bounded by $y = \sqrt{x}, y = 2 - x$ and y = 0



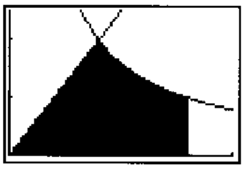
A type II integral will be only one. $\int_0^1 \int_{y^2}^{2-y} ydxdy = \int_0^1 (2y - y^2 - y^3)dy = \frac{5}{12}$ whereas a type I integral will be two double integrals $\int_0^1 \int_0^{\sqrt{x}} ydydx + \int_1^2 \int_0^{2-x} ydydx.$

<u>eg 9</u> Find $\int_R (2y)dA$ where *R* is bounded by $y = x^2$ and $y = \sqrt{x}$. If we use a Type I integral, the region is bounded above by $y(x) = \sqrt{x}$ and below by $y(x) = x^2$ in $0 < x < 1$. The integral then becomes
$\int_0^1 \int_{x^2}^{\sqrt{x}} (2y)dydx = \int_0^1 (y^2)|_{x^2}^{\sqrt{x}} dx = \int_0^1 (x - x^4)dx = \left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{10}.$

If we use a Type II integral, the region is bounded at the right by $x(y) = \sqrt{y}$ and at the left by

$x(y) = y^2$ in $0 < y < 1$. The integral then becomes $\int_0^1 \int_{y^2}^{\sqrt{y}} (2y)dxdy = \int_0^1 (2xy)|_{y^2}^{\sqrt{y}} dy =$
$\int_0^1 (2y^{3/2} - 2y^3)dy = \left(\frac{4}{5} - \frac{1}{2}\right) = \frac{3}{10}$.

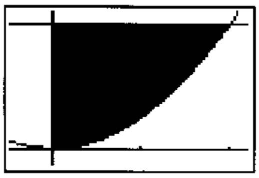
eg 10 Find $\int_R x^2 dA$ where R is bounded by $xy = 4$ and $y = x$, $y = 0$, $x = 4$.



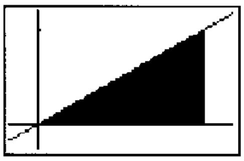
If we use a type I integral, we need two double integrals.
$\int_0^2 \int_0^x x^2 dydx + \int_2^4 \int_0^{4/x} x^2\, dydx = \int_0^2 x^3\, dx + \int_2^4 4xdx = 28.$
If we use a type II integral, we need also two double integrals.
$\int_0^1 \int_y^4 x^2\, dxdy + \int_1^2 \int_y^{4/y} x^2\, dxdy = 28.$

Reversing the Order of Integration
Sometimes it is easier to reverse the order of integration to make the integration easier.

eg 11 To evaluate $\int_0^2 \int_{x^2}^4 \frac{x}{\sqrt{1+y^2}} dydx$, we need the trig substitution $y = tan\,(\theta)$. The integral then becomes
$\int_0^2 \int_{Arctan(x^2)}^{\arctan(4)} [xsec(\theta)]d\theta dx \int_0^2 [xln(\sec(\theta) + \tan(\theta))\,|_{\theta=Arctan(x^2)}^{\theta=Arctan(4)} dx =$
$\int_0^2 xln\left(\sqrt{17} + 4 - \sqrt{x^4+1} - x^2\right) dx$. This type I integral is very complicated. A simpler integral is found if we change the order of integration and make the integral a Type II integral.
If we plot the region of integration, we obtain $\int_0^2 \int_{x^2}^4 \frac{x}{\sqrt{1+y^2}} dydx = \int_0^4 \int_0^{\sqrt{y}} \frac{x}{\sqrt{1+y^2}} dxdy = \int_0^4 \frac{y}{2\sqrt{1+y^2}} dy =$
$\frac{1}{2}\left(\sqrt{17} - 1\right)$.



eg 12 $\int_0^1 \int_{3y}^3 e^{x^2} dxdy$ cannot be evaluated as it is. If we change the order of integration, by plotting the region of integration, we obtain



$\int_0^3 \int_0^{x/3} e^{x^2} dydx = \int_0^3 \frac{xe^{x^2}}{3} dx = \frac{(e^9 - 1)}{6}$.

eg 13 Evaluate $\int\int_R \frac{1}{1+x^2} dA$ where R is the triangle (0, 0), (0, 1), (1, 1). If we use a type I integral,

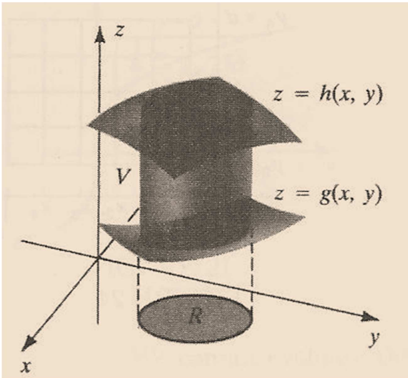
$\int_0^1 \int_x^1 \frac{1}{1+x^2}\, dydx = \int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2}\right) dx = tan^{-1}(1) - \frac{ln(2)}{2}.$

If we use a type II integral, $\int_0^1 \int_0^y \frac{1}{1+x^2} dxdy = \int_0^1 tan^{-1}(y)dy$. If we then integrate by parts, we obtain

$\int_0^1 tan^{-1}(y)dy = y\, tan^{-1}(y)|_0^1 - \int_0^1 \frac{y}{1+y^2} dy = \frac{\pi}{4} - \frac{ln(2)}{2}.$

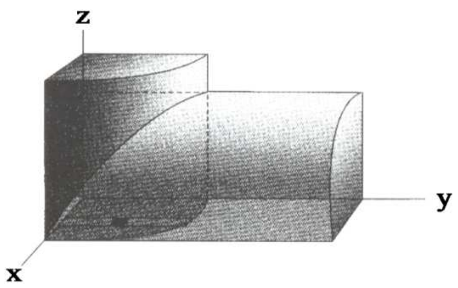
Volumes Between Two Surfaces

The volume between the surface $z_{bottom} = g(x, y)$ and $z_{top} = h(x, y)$ can be given by $\int\int_R (z_{top} - z_{bottom}) dA$, where *dA* is either *dxdy* or *dydx*.

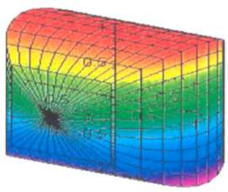


eg 14 Find the volume bounded by the plane *x* + *y* + *z* = 1 in the first octant.
Since $z_{top} = 1 - x - y$ and $z_{bottom} = 0$, $\int_0^1 \int_x^{1-x} (1 - x - y)dydx = \frac{1}{6}$.

eg 15 Find the volume bounded by the circular cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.



First octant view of the volume



Volume

If we let *R* be the circle $x^2 + y^2 = 1$ with $z_{top} = \sqrt{1 - x^2}$ and $z_{bottom} = -\sqrt{1 - x^2}$, the integral for the volume

becomes $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\sqrt{1 - x^2} - \left(-\sqrt{1 - x^2}\right)\right) dydx = \int_{-1}^1 \left(2\sqrt{1 - x^2}\right) y|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = 4 \int_{-1}^1 (1 - x^2)\, dx =$

$8 \int_0^1 (1 - x^2)\, dx = \frac{16}{3}.$

eg 16 Find the volume in the first octant of the paraboloid $f(x, y) = 2 - x^2 - y^2$.

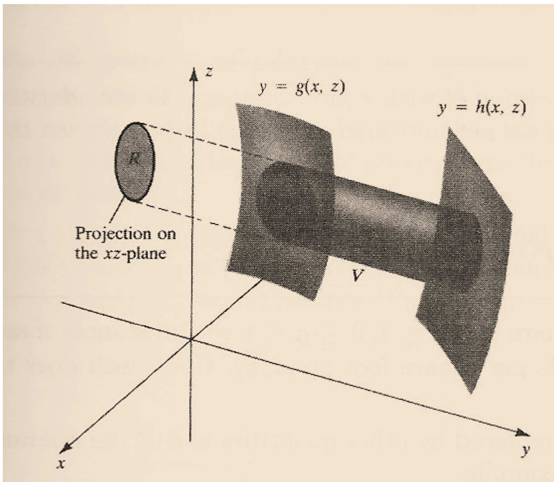
$\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}(2-x^2-y^2)\,dydx = \int_0^{\sqrt{2}}\left((2-x^2)\sqrt{2-x^2} - \frac{(\sqrt{2-x^2})^3}{3}\right)dx = \frac{2}{3}\int_0^{\sqrt{2}}(\sqrt{2-x^2})^3\,dx.$

If we apply the trig substitution,

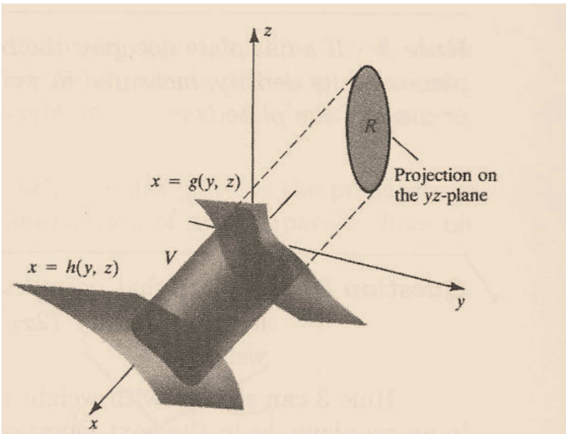
$\frac{8}{3}\int_0^{\pi/2}cos^4 d\theta = \int_0^{\pi/2}\frac{8}{3}(\frac{1+cos\,2\theta}{2})^2 d\theta = \int_0^{\pi/2}\frac{2}{3}\left(1 + 2\,cos\,2\theta + \left(\frac{1+cos\,4\theta}{2}\right)\right)d\theta =$

$\frac{2}{3}\left(\frac{3}{2}\theta + sin\,2\theta + \frac{sin\,4\theta}{8}\right)]|_0^{\pi/2} = \pi/2.$

We can also find volumes of solids by integrating over the projection on the *xz* plane.
If the region R is on the *xz* plane, the integral becomes $\int\int_R (h(x,z) - g(x,z))dA$, where *dA* is either *dxdz* or *dzdx*. See picture below. Refer to homework problem 13.



We can also find volumes of solids by integrating over the projection on the *yz* plane.
If the region R is on the *yz* plane, the integral becomes $\int\int_R (h(y,z) - g(y,z))dA$, where *dA* is either *dydz* or *dzdy*. See picture below. Refer to homework problem 14.



Homework 16.2

1. $\int_0^1 \int_{-1}^1 (1 + x^2y^2\, dxdy$ Ans: 20/9

2. $\int_0^{ln2} \int_0^1 xe^{xy}\, dxdy$ Ans: $\frac{1-ln2}{ln2}$

3. $\int_1^3 \int_0^1 e^{x+y}\, dxdy$ Ans: $e - e^2 - e^3 + e^4$

4. $\int_0^1 \int_{\sqrt{x}}^x x^2y\, dydx$ Ans: -1/40

5. $\int_0^1 \int_0^\pi xysinx\, dxdy$ Ans: $\pi/2$

6. Find the average value of $f(x,y)=x^2y^2$ in the rectangle $0\le x \le 1, 0\le y \le 2$.
   Ans: 4/9

7. Evaluate $\int_R f(x,y)dydx$ for $f(x,y)=10x^4y$ where R is the triangle with vertices (0,0),(0,2)(1,1).
   Ans: 2/3

8. Evaluate $\int_R f(x,y)dxdy$ for $f(x,y)=4x^3y$ where R is bounded by $y = x^2$ and $y = 2x$.
   Ans: 64/3

9. Evaluate $\int_R f(x,y)dxdy$ for $f(x,y)= y^2\cos(x)$ where R is bounded by $x = y^3$ and the lines $y = 0$ and $x=\pi$.
   Ans: -2/3

10. Reverse the order of integration to evaluate $\int_0^4 \int_y^4 e^{-x^2}\, dxdy$
    Ans: $\frac{1}{2}(1 - e^{-16})$

11. Find the volume of the tetrahedron bounded by coordinate planes and the plane with $x_{\text{int}} =1$; $y_{\text{int}} = 1$ and $z_{\text{int}} =1$. Ans: 1/6

12. Find the volume of the solid bounded by the planes x=0, y=0, $2x+2y+z=2$and $4x+4y-z = 4$. Ans: 1

13. Find the volume of the solid bounded by $y= x^2+z^2$ and the plane $y =-3$ for (x,z) in the rectangle x=0, x=4, z=0,z=5. Ans: 1000/3

14. Find the volume of the solid between the surfaces $x= -z^2$ and $x= z^2 +2$ for (y,z) in the rectangle y=0, y=4, z=0,z=1. Ans: 32/3

16.4 Double Integrals in Polar Coordinates

Any point (x, y) in $\Re^2$ can be expressed in Polar Coordinates as (r, θ), where $r = \sqrt{x^2 + y^2}$; $\tan(\theta) = \frac{y}{x}$; $x = r\cos(\theta)$; $y = r\sin(\theta)$ are the transformations equations. The area of the polar function $r(\theta)$, with a single integral, is given by $Area = \frac{1}{2}\int_a^b r^2 d\theta$.

eg 17 Find the area of the cardioid $r = 1 + \cos\theta$.

$Area = \frac{1}{2}\int_0^{2\pi}(1 + \cos\theta)^2\, d\theta. = \frac{1}{2}\int_0^{2\pi}(1 + 2\cos\theta + \cos^2\theta)d\theta. = \frac{1}{2}\left(\theta + 2\sin(\theta) + \frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right)|_0^{2\pi} = \frac{3\pi}{2}$.

The area of the polar function $r(\theta)$, with a double integral, is given by

$Area = \int_R f(x, y)dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} rdrd\theta$ where the element of area $dA = rdrd\theta$.

eg 18 Find the area, using double integrals, for the previous problem.

$Area = \int_0^{2\pi}\int_0^{1+\cos\theta} rdrd\theta = \int_0^{2\pi}\frac{(1+\cos\theta)^2}{2}d\theta = \frac{3\pi}{2}$.

eg 19 Find the area outside the circle $r = 2$ and inside the cardioid $r = 2 + 2\cos\theta$

$Area = \int_{-\pi/2}^{\pi/2}\int_2^{2+2\cos\theta} rdrd\theta = 2\int_0^{\pi/2}\left(\frac{(2+2\cos\theta)^2}{2} - \frac{2^2}{2}\right)d\theta = \int_0^{\pi/2}(8\cos(\theta) + 4\cos^2(\theta))d\theta =.$

$$\int_0^{\pi/2}(8\cos(\theta) + 2(1 + \cos(2\theta)))d\theta = 8 + \pi$$

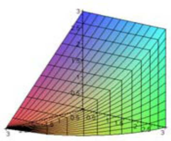
eg 20 Evaluate the volume in the first octant of the paraboloid $f(x, y) = 2 - x^2 - y^2$.

If we use polar coordinates, $V = \int_0^{\pi/2}\int_0^{\sqrt{2}}(2 - r^2)rdrd\theta = \int_0^{\pi/2} r^2 - \frac{r^4}{4}|_0^{\sqrt{2}}d\theta = \int_0^{\pi/2} 1d\theta = \pi/2$.

eg 21 Evaluate $\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2 + y^2}\,dydx$.

Since there is circular symmetry, we can change the integral to polar coordinates. Since $y = \sqrt{2x - x^2}$ or $x^2 + y^2 = 2x$, is the semicircle with center (1, 0) and radius 1, $r = 2\cos(\theta)$ in $0 < \theta < \pi/2$. The integral becomes $\int_0^{\pi/2}\int_0^{2\cos(\theta)} rrdrd\theta = \int_0^{\pi/2}\frac{8(\cos\theta)^3}{3}d\theta = \frac{16}{9}$.

eg 22 Find the volume in the first octant between $x^2 + y^2 = 9$ and $x + z = 3$.



In Cartesian Coordinates the integral becomes $\int_0^3 \int_0^{\sqrt{9-x^2}} (3-x)dydx$. If we change to polar coordinates, we have $\int_0^{\frac{\pi}{2}} \int_0^3 (3 - r\cos\theta) r dr d\theta = \frac{27\pi}{4} - 9$.

Homework: 16.4

1. Use polar coordinates to evaluate the $\int\int_R y dxdy$ where R is the region bonded by the upper half of the cardioid $r = 1+\cos(\theta)$ and the x-axis. Ans: 4/3
2. What is the area inside one leaf of the rose r = 2 cos (3θ)? Ans: π/3
3. Use polar coordinates to evaluate the $\int\int_R xy dxdy$ where R is the region bonded by the semicircle $y = \sqrt{x - x^2}$ and the x-axis. Ans: 1/24
4. Evaluate the $\int\int_R (x^2+y^2)^{-2} dxdy$ with $R = \{(x,y): 2 \le x^2 + y^2 \le 4\}$. Ans: π/4
5. Find the value of $\int_{3\pi/4}^{4\pi/3} \int_0^{-5\sec\theta} r^3 \sin^2\theta \, dr \, d\theta$ by first switching it from polar to rectangular coordinates. Ans: $\frac{625}{12}(3\sqrt{3}+1)$
6. The integrals (a) $\int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} \frac{r^2}{1+r\sin\theta} \, dr \, d\theta$ and (b) $\int_{\pi/4}^{3\pi/4} \int_0^{4\csc\theta} r^5 \sin^2\theta \, dr \, d\theta$ are given in polar coordinates. Rewrite them as iterated integrals in rectangular coordinates.

Ans: $\int_0^1 \int_1^{1+\sqrt{1-y^2}} \frac{\sqrt{x^2+y^2}}{1+y} \, dx \, dy = \int_1^2 \int_0^{\sqrt{1-(x-1)^2}} \frac{\sqrt{x^2+y^2}}{1+y} \, dy \, dx$, $\int_0^4 \int_{-y}^{y} (x^2+y^2) y^2 \, dx \, dy$

Applications of double integrals:

Average Value of a Function

If $f(x, y)$ is piecewise continuous in a bounded region R with piecewise smooth boundary, then $f_{ave} = \frac{1}{area} \int_R f(x,y) dA = \frac{\iint f(x,y) dA}{\iint dA}$.

<u>**eg 23**</u> Find the average value of $f(x, y) = 2ye^{y^2}$ in $0 < x < 1; 0 < y < \sqrt{x}$.

$f_{ave} = \frac{\int_0^1 \int_0^{\sqrt{x}} 2ye^{y^2} dydx}{\int_0^1 \int_0^{\sqrt{x}} dydx} = \frac{(e-2)}{2/3}$.

Centroids (Center of Mass)

The moment M (also called first moment) is defined as $M = mr$ where m is the mass and r is moment arm (distance from the particle to the axis). For a discrete system of particles of mass m_i: In one dimension, the moment about the origin is given by $M = \sum_i m_i x_i$, so the center of mass will be

$\bar{x} = M / \sum_i m_i = \sum_i m_i x_i / \sum_i m_i$

In two dimensions, the moment about the axis is given by $M_x = \sum_i m_i\, y_i$ and $M_y = \sum_i m_i\, x_i$

so the center of mass will be $\bar{x} = \frac{M_y}{m} = \sum_i m_i\, x_i / \sum_i m_i$ and $\bar{y} = \frac{M_x}{m} = \sum_i m_i\, y_i / \sum_i m_i$
For a one dimension continuous system with mass density (linear) ρ, the center of mass will be

$\bar{x} = \frac{M_x}{m} = \frac{\int \rho x dx}{\int \rho dx}$. If the mass density is uniform (constant), $\bar{x} = \frac{\int \rho x dx}{\int \rho dx} = \frac{\int x dx}{\int dx}$.
For a two dimensions thin plate with a uniform (constant) mass (area) density ρ bounded by the function $f(x) - g(x)$, the center of mass will be

$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b \rho x[f(x)-g(x)]dx}{\int \rho(f(x)-g(x))dx}$ and $\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b \rho[f^2(x)-g^2(x)]dx}{\int \rho(f(x)-g(x))dx}$.

<u>**eg 24**</u> Find the centroid of the triangular lamina bounded by $y = 2x$, $y = 0$, $x = 1$ with uniform density $3gm/cm^2$.
$\bar{x} = \frac{M_y}{m} = \frac{\int_0^1 3x[2x-0]dx}{\int_0^1 3(2x)dx} = \frac{2}{3}$; and $\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_0^1 3[(2x)^2-0^2]dx}{\int_0^1 3(2x)dx} = \frac{2}{3}$

For a two dimensional thin plate with non-uniform (variable) mass density $\rho(x, y)$.
$\bar{x} = \frac{M_y}{m} = \frac{\int\int x\rho(x,y)dA}{\int\int \rho dA}$ and $\bar{y} = \frac{M_x}{m} = \frac{\int\int y\rho(x,y)dA}{\int\int \rho dA}$.

<u>**eg 25**</u> Find the centroid of the triangular lamina with vertices (0, 0), (1, 0) and (0, 1) with mass density $\rho(x, y) = xy\ kg/m^2$.

$\bar{x} = \frac{M_y}{m} = \frac{\int_0^1\int_0^{1-x} x^2 y\, dydx}{\int_0^1\int_0^{1-x} xy\, dydx} = \frac{2}{5}$ and $\bar{y} = \frac{M_x}{m} = \frac{\int_0^1\int_0^{1-x} xy^2\, dydx}{\int_0^1\int_0^{1-x} xy\, dydx} = \frac{2}{5}$.

<u>**eg 26**</u> Find the centroid of a lamina shaped in the form of a quarter circle or radius 1 with density proportional to the distance from the center of a circle. Because of the symmetry, $\bar{x} = \bar{y}$. Since $\rho = k\sqrt{x^2 + y^2}$, if we use polar coordinates,
$\bar{x} = \bar{y} = \frac{\int\int x\rho(x,y)dA}{\int\int \rho dA} = \frac{\int\int k\sqrt{x^2+y^2}\, dA}{\int\int k\sqrt{x^2+y^2}\, dA} = \frac{\int_0^{\pi/2}\int_0^1 r\cos(\theta) rrdrd\theta}{\int_0^{\pi/2}\int_0^1 rrdrd\theta} = \frac{\int_0^{\pi/2}\frac{\cos(\theta)}{4}d\theta}{\int_0^{\pi/2}\frac{1}{3}d\theta} = \frac{3}{2\pi}$.

Moments of Inertia of Plane Area

The moment of inertia I (also called second moment) is defined as $M = mr^2$ where m is the mass and r is moment arm (distance from the particle to the axis). The moment of inertia of a plane area about an axis relates the angular acceleration about the axis to the force twisting the plane area (torque). The moment of inertia of a plane lamina of density $\rho(x, y)$ about an axis is defined as $\int\int p^2\, dm$ where p is the perpendicular distance from a point (x, y) to the axis and $dm = \rho(x, y)dA$ where $\rho(x, y)$ is the area density, dA is the element of area and dm is the element of mass.
The moment of inertia about the x-axis will be $I_x = \int\int y^2\, \rho(x, y)dA$.
The moment of inertia about the y-axis will be $I_y = \int\int x^2\, \rho(x, y)dA$;
The moment of inertia about the origin (polar moment) will be $I_0 = I_x + I_y = \int\int (x^2 + y^2)\rho(x, y)dA$.

eg 27 Find I_x, I_y and I_0 of a lamina of constant density k, bounded by $x = y^2$, $x = -y^2$ in $-1 \le y \le 1$.

$I_x = \int_{-1}^{1}\int_{-y^2}^{y^2} y^2\, kdxdy = k\int_{-1}^{1} 2y^4\, dy = k\frac{4}{5}$; $I_y = k\int_{-1}^{1}\int_{-y^2}^{y^2} x^2\, dxdy = k\int_{-1}^{1}\frac{2}{3}y^6 dy = k\frac{4}{21}$; $I_0 = k\frac{104}{105}$.

eg. Find the moment of inertia about the origin on a lamina bounded by circle $x^2 + y^2 = 4$ with density $\rho(x, y) = k$ a constant.

$I_0 = I_x + I_y = \int\int (x^2 + y^2)\rho(x, y)dA. = \int_0^{2\pi}\int_0^2 r^2 rdrd\theta = 8k\pi.$

Area of a Surface Described by $z = f(x, y)$

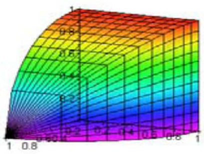
The area of the surface $z = f(x, y)$, (x, y) in D, where f, f_x and f_y are continuous is given by

$A(s) = \int\int_D \sqrt{f_x^2(x, y) + f_y^2(x, y) + 1}dA.$

eg 28 Calculate the surface area of the portion of the surface $z = xy$ for $D : x^2 + y^2 \le 9$.

$A(s) = \int\int_D\sqrt{y^2 + x^2 + 1}\, dA = \int_0^{2\pi}\int_0^3 \sqrt{r^2 + 1}\; rdrd\theta = \frac{2\pi}{3}(r^2 + 1)^{\frac{3}{2}} = \frac{2\pi}{3}(10^{\frac{3}{2}} - 1).$

eg 29 Find the surface area of the volume in the first octant bounded by the circular cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.



The area of the volume in the x-y $dydx = \frac{\pi}{4} + \frac{\pi}{4} + 1 + \int_0^1 dx = 2 + \frac{\pi}{2}$ square units.

Homework

1. Find the average value of $f(x, y)=x^2y^2$ in the rectangle $0\le x \le 1$, $0\le y \le 2$.
   Ans: 4/9
2. A plate in the xy-plane with distances measured in ft. occupies the region bounded by the parabola $y=x - y^2$ and the x-axis and its density at (x, y) is $6x^2$ lb/ft^2. (a) How much does the plate weigh? (b) Where is the center of gravity? Ans: (a) 3/10 lb, (b) (2/3,2/21)
3. Find the centroid of the quarter circle lying between $y = \sqrt{1 - x^2}$ and the y-axis for $0\le x \le 1$ with density ρ=1. Ans: $(\frac{4}{3\pi}, \frac{4}{3\pi})$
4. A flat disk occupies the disk $x^2 - 2x + y^2 \le 0$ in the xy –plane with distances measured in inches and, and its area density at (x,y) is $\rho = 5\sqrt{x^2 + y^2}$ ounces per square inch. Where is the center of gravity?
   Ans: (6/5,9/20)

16.3 Triple Integrals

Let $f(x, y, z)$ be a continuous function in a regions V in $\Re^3$. The triple integral over R is $\int\int\int_V f(x,y,z)dV$ where dV = is an element of volume. If $f(x, y, z) = 1$, the triple integral will give the volume of the region V.

Iterated Integrals over Rectangular Regions

If I is an iterated integrals over rectangular region, the integration can be switched.

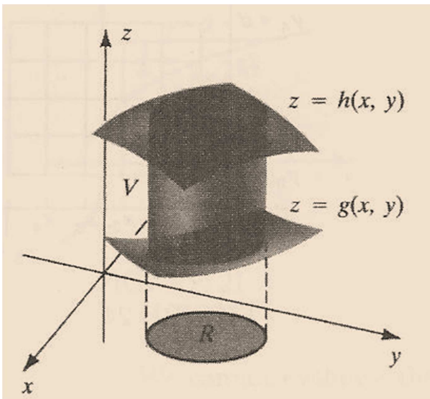
$$I = \int_a^b \int_c^d \int_e^f f(x,y,z)dxdydz = \int_c^d \int_e^f \int_a^b f(x,y)dydzdx = \int_e^f \int_a^b \int_c^d f(x,y)dzdxdy \ldots$$

eg 30 $\int_{\frac{1}{3}}^{\frac{1}{2}} \int_0^{\pi} \int_0^1 zxsin(xy)dzdydx = \frac{1}{12} + \frac{\sqrt{3-2}}{4\pi}$ cu units.

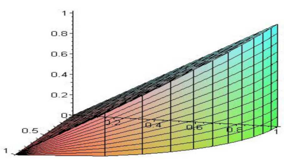
Iterated Integrals over General Regions

Type I

If the region of integration in $\Re^3$ is bounded above by the function $h(x, y)$ and below by the function $g(x, y)$ in the area A in the xy plane, the triple integral will be $\int_V f(x,y,z)dV = \int\int\left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)dz\right)dA$, where dA is $dxdy$ or $dydx$. See figure below.

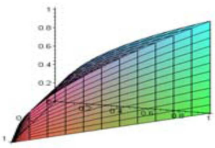


eg 31 Find $\int\int\int_V xdV$ where R is the region bounded by the plane $y = z$, and the cylinder $x^2 + y^2 = 1$ in the first octant.



Since the region is bounded above by $y = z$ and below by the xy plane, the integral becomes $\int_0^1 \int_0^{\sqrt{1-y^2}} \left(\int_0^y x\, dz\right) dxdy = \frac{1}{8}$.

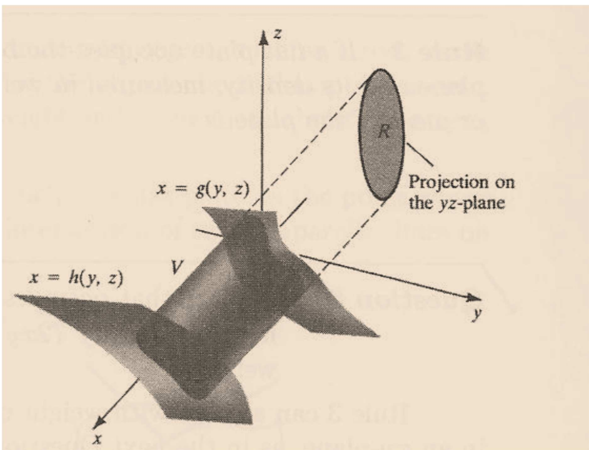
eg 32 Find the volume bounded by the parabolic surface $z = \sqrt{y}$ the plane $x + y = 1$ and the x-y plane.



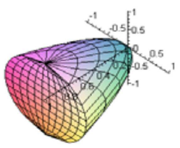
$$V = \int_0^1 \int_0^{1-x} \int_0^{\sqrt{y}} 1 dzdydx = \int_0^1 \int_0^{1-x} \sqrt{y}\ dydx = \int_0^1 \frac{2}{3}(1-x)^{3/2}dx = \frac{2}{3}\frac{2}{5}(1-x)^{5/2}|_1^0 = \frac{4}{15}.$$

Type II

If the region of integration in $\Re^3$ is bounded in the front by the function $g_2(y, z)$ and in the back by the function $g_1(y, z)$ in the area A in the yz plane, the triple integral will be
$\int\int\int_V f(x,y,z)dV = \int\int\left(\int_{g(y,z)}^{h(y,z)} f(x,y,z)dx\right)dA$, where $d\,A$ is $dydz$ or $dzdy$. See figure below.



eg. 33 Set $\int\int\int_V xdV$ where V is the region bounded by the plane $x = 1$, and the paraboloid $x = y^2 + z^2$.

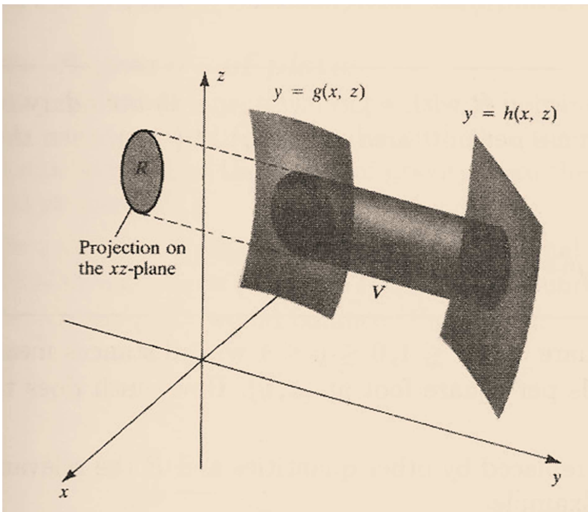


Since the region is bounded in the back by $x = y^2 + z^2$ and in the front by the $x = 1$ plane, the integral becomes
$\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\left(\int_{y^2+z^2}^{1} xdx\right)dydz.$

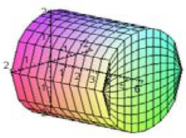
A simple way of solving this integral is defining a polar coordinate system in the y-z plane such that
$y = r\,cos(\theta), z = r\,sin(\theta)$ with $r^2 = x^2 + y^2$ and $tan(\theta) = \frac{z}{y}$.
If we do so, the integral becomes $\int_0^{2\pi}\int_0^1\left(\int_{r^2}^1 x\,dx\right)rdrd\theta = \pi/3$.

Type III

If the region of integration in $\Re^3$ is bounded in the right by the function g_2 (x, z) and in the left by the function g_1 (x, z) in the area A in the yz plane, the triple integral will be
$\int\int\int_V f(x,y,z)dV = \int\int\left(\int_{g(x,z)}^{h(x,z)} f(x,y,z)dy\right)dA$, where dA is $dxdz$ or $dzdx$



eg 34 Find $\int\int\int_V y dV$ where V is the region bounded by the cylinder $x^2 + z^2 = 4$, and the planes $y = 0$ and $y = 6$.



Since the region is bounded in the right by the plane $y = 0$ and in the right by the $y = 6$ plane, the integral becomes $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(\int_0^6 y\,dy\right)dzdx = 72\pi.$
A simple way of solving this integral is defining a polar coordinate system in the x-z plane such that $x = r\cos(\theta),\;\; z = r\sin(\theta)$ with $r^2 = x^2 + z^2$ and $\tan(\theta) = \frac{z}{x}$.
If we do so, the integral becomes $\int_0^{2\pi}\int_0^2\left(\int_0^6 ydy\right)rdrd\theta = 72\pi.$

Average Value of a Function

If $f(x, y, z)$ is piecewise continuous is a bounded region with piecewise smooth boundary, then
$f_{ave} = \frac{1}{volume}\int\int\int f(x,y,z)dV = \frac{\int\int\int f(x,y,z)dV}{\int\int\int dV}$.

eg 35 Find the average value of $f(x, y, z) = xyz$ in $0 < x < 1;\ 0 < y < x^3;\ 0 < z < 8$.
As a Type I integral, $f_{ave} = \frac{\int_0^1\int_0^{x^3}\int_0^8 xyz\,dzdydx}{\int_0^1\int_0^{x^3}\int_0^8 dzdydx} = \frac{2}{2} = 1$

Centroids

Definition: If $\rho(x,y,z)$ is a mass (weight) density, the mass (weight) is given by $m = \int\int\int_v \rho(x,y,z)\, dV$.

The centroid will be $\bar{x} = \frac{M_{yz}}{m} = \frac{\int\int\int x\rho\,(x,y,z)dV}{\int\int\int \rho dV}$, $\bar{y} = \frac{M_{xz}}{m} = \frac{\int\int\int y\rho\,(x,y,z)dV}{\int\int\int \rho dV}$, $\bar{z} = \frac{M_{xy}}{m} = \frac{\int\int\int z\rho\,(x,y,z)dV}{\int\int\int \rho dV}$

Moments of Inertia of a Solid Body

The moment of inertia of a solid body about an axis relates the angular acceleration about the axis to the force twisting the solid (torque). Moment of inertia of a solid body of density $\rho(x,y)$ about an axis is defined as $\int\int\int p^2\, dm$, where p is the perpendicular distance from a point (x,y,z) to the axis and $dm = \rho(x,y,z)dV$ where $\rho(x,y,z)$ is the volume density, dV is the element of volume and dm is the element of mass.
The moment of inertia about the x-axis will be $I_x = \int\int\int(y^2+z^2)\,\rho\,(x,y,z)dV$.
The moment of inertia about the y-axis will be $I_y = \int\int\int(x^2+z^2)\,\rho\,(x,y,z)dV$.
The moment of inertia about the y-axis will be $I_z = \int\int\int(x^2+y^2)\,\rho\,(x,y,z)dV$.

Homework: 16.3

1. Express $\int\int\int_V f(x,y,z)dV$ as an integral where V is the solid bounded by $z = 0$, $z = 1-y^2$, $x = -2$ and $x = 0$ as a Type I integral.
   Ans $\int_{-2}^{0}\int_{-1}^{1}\int_{0}^{1-y^2} f dzdydx$
2. What is the value of $\int\int\int_V F\, dx\, dy\, dz$ if F is the constant function $F(x, y, z) = 7$ and V is a bounded solid with piecewise smooth boundary whose volume is 10? (b) What is the average value of F in V for the function and solid part (a)? Ans: 70,7
3. A block occupies the region bounded by $x = -2$, $x = 2$, $y = -2$, $y = 2$, $z = 1$, and $z = 2$ in xyz-space with distances measured in meters and its density at (x, y, z) is $x^2e^y \sin z$ kilograms per cubic meter. What is its mass? Ans: $\frac{16}{3}(\cos(1)-\cos(2))(e^2-e^{-2})$ kg.
4. A solid V bounded by $z = x^2 + 1$, $z = -y^2 - 1$, $x = 0$, $x = 2$, $y = -1$, and $y = 1$ in xyz-space with distances measured in feet contains electrical charges with density $8xy^2$ coulombs per cubic foot at (x, y, z). What is the overall net charge in V? Ans:736/15

5. What is the average value of $f = xy^3z^7$ in the box V bounded by $x = 0$, $x = 2$, $y = 0$, $y = 2$, $z = 0$ and $z = 2$? Ans:32
6. Describe the solids of integration that lead to the iterated integrals:

$$\int_{x=0}^{x=2}\int_{y=0}^{y=\sqrt{4-x^2}}\int_{z=0}^{z=\sqrt{4-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

Ans: one fourth of an upper hemisphere centered at the origin with radius 2, with $x \geq 0$ and $y \geq 0$.

7. Describe the solids of integration that lead to the iterated integrals:

$$\int_{y=0}^{y=1}\int_{z=0}^{z=2-2y}\int_{x=0}^{x=4-4y-2z} f(x,y,z)dxdzdy$$

Ans: The tetrahedron bounded by the plane $x - 4y - 2z = 4$ and the coordinate planes.

8. Evaluate $\int\int\int_V x^2y^2z\, dV$ with V bounded by $0 < x < 1$, $0 < z < x^2 - y^2$. Ans: k=4/525

9. Find k such that $\int\int\int_V x^2z\, dV = \dfrac{1}{60}$ where V bounded by $z = 0$, $z = x$, $x = 0$, $y = 0$, and $x + y = k$. Ans: k=1

10. What is the centroid of the cylinder V = $\{(x, y, z): x^2 + y^2 \le 4, 0 \le z \le 1\}$ in *xyz*-space with distances measured in feet if its density at (x, y, z) is $\cos z$ pounds per square foot?

    Ans: $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \dfrac{\sin(1) + \cos(1) - 1}{\sin(1)})$

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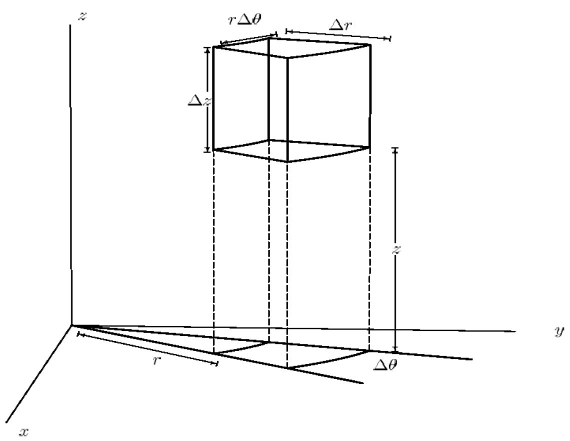
16.5 Integrals in Cylindrical and Spherical Coordinates

Cylindrical Coordinate System

Any point (x, y, z) in $\Re^3$ can be expressed in Cylindrical Coordinates as (r, θ, z) where $r = \sqrt{x^2 + y^2};\ tan(\theta) = \frac{y}{x}; z = z; x = r\, cos(\theta); y = r\, sin(\theta)$ are the transformation equations, for $0 \le r < \infty,\ 0 \le \theta \le 2\pi, -\infty < z < \infty$.

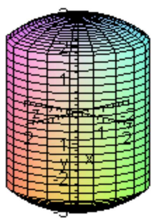
If c is a constant, $r = c$ is a cylinder; $\theta = c$ is a plane; $z = c$ is a plane.



A triple integral in cylindrical coordinates is given by $\int_R f(x, y, z)dV$

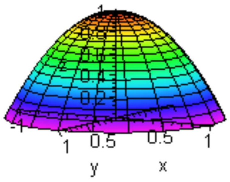
$\int_{z_1}^{z_2}\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos(\theta),\ r\sin(\theta), z)\ rdzdrd\theta$ where the element of volume $dV = rdzdrd\theta$.

eg 36 Use a triple integral to find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the sphere $x^2 + y^2 + z^2 = 9$.



In Cartesian Coordinates the volume is $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} dzdydx$. If we change to cylindrical coordinates, we have $\int_0^{2\pi}\int_0^{2}\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} rdzdrd\theta = \frac{4\pi}{3}\left(27-5\sqrt{5}\right)\ cu\ units.$

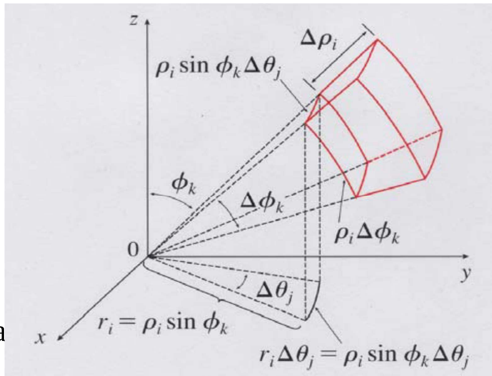
<u>eg 37</u> Compute the triple integral of $f(x, y, z) = \sqrt{x^2+y^2}$ bounded by the paraboloid $x^2 + y^2 + z = 1$ and the plane $z = 0$.



$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-x^2-y^2}\sqrt{x^2+y^2}\, dzdydx = \int_0^{2\pi}\int_0^{1}\int_0^{1-r^2} r^2\, dzdrd\theta = \int_0^{2\pi}\int_0^{1} r^2\,(1-r^2)drd\theta =$ $2\pi\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{4\pi}{15}.$

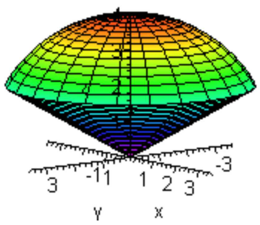
Spherical Coordinate System

Any point (x, y, z) in $\Re^3$ can be expressed in Spherical Coordinates as (ρ, θ, ϕ) where
$\rho = \sqrt{x^2+y^2+z^2};\ tan(\theta) = \frac{y}{x};\ cos\,\phi = \frac{z}{\rho} = \frac{z}{\sqrt{x^2+y^2+z^2}};$
$x = \rho\, sin(\phi)\cos(\theta)\,; y = \rho sin\,(\phi)\, sin(\theta)\,; z = \rho cos(\phi)$ are the transformations equations for
$0 \le \rho < \infty,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$
If c is a constant, $\rho = c$ is a sphere; $\theta = c$ is a plane; $\phi = c$ is a cone.



A triple integra ... $f(x,y,z)dV =$
$\int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2}\int_{\rho_1}^{\rho_2} f(\rho\, sin(\phi)\cos(\theta), \rho\, sin(\phi)\, sin(\theta), \rho\, cos(\phi))\rho^2\, sin(\phi)d\rho d\theta d\phi$ where the element of volume $dV = \rho^2\, sin(\phi)d\rho d\theta d\phi.$

<u>eg 38</u> Find the volume bounded by the cone $\phi = \frac{\pi}{3}$ and the sphere $\rho = 4$.

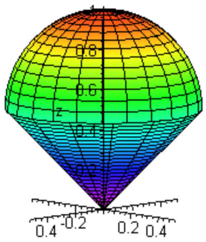


$V = \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_0^{4} \rho^2 \sin(\phi) d\rho d\theta d\phi = \frac{64\pi}{3}\ cu\ units.$

eg 39 Evaluate $\int\int\int e^{(x^2+y^2+z^2)^{3/2}} dV$ in the unit ball.

$V = \int_0^{\pi} \int_0^{2\pi} \int_0^{1} e^{\rho^3} \rho^2 sin(\phi) d\rho d\theta d\phi = \frac{4\pi}{3}(e-1)$.

eg 40 Find the volume bounded by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = z$.



Since $\rho^2 = \rho\cos\phi$, $0 < \rho < \cos\phi$, and $\rho\cos\phi = \sqrt{(\rho\, sin(\phi)\, cos(\theta))^2 + (\rho\, sin(\phi)\, sin(\theta))^2} = \rho\, sin(\phi)$, we can say that the cone goes from, $0 < \phi < \frac{\pi}{4}$.

So $= \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\cos\phi} \rho^2 sin(\phi) d\rho d\theta d\phi = \frac{2\pi}{3} \int_0^{\pi/4} (cos\,\phi)^3 sin(\phi) d\phi = \frac{\pi}{8}$.

Homework 16.5

1. Describe the solid given by:
a) $0 \le \theta \le \pi/2,\ \ r \le z \le 2;$ Ans: The first octant section of the cone $\sqrt{x^2+y^2} = z$ and the plane z = 2.
b) $0 \le \rho \le 3, \pi/2 \le \phi \le \pi;$ Ans: The bottom half of the sphere of radius 3.
c) $0 \le \phi \le \pi/4,\ \ \rho \le 2;$ Ans: A snow cone

2. Find the coordinate of the point $(\rho, \phi, \theta) = (\sqrt{2}, \pi/4, \pi/2)$
a) in the cylindrical (r, θ, z) coordinate systems. Ans: $[(1, \pi/2, 1)]$
b) in the Cartesian (*x*, *y*, *z*) coordinate systems. Ans: [(0, 1, 1)]

3. Change to the other two coordinate system and sketch
a) $tan\,\phi = 1.$ Ans: $[x^2+y^2 = z^2;\ r^2 = z^2\ cone]$
b) $r = 2\,cos(\theta).$ Ans: $x^2+y^2 = 2x;\ \rho\, sin(\phi) = 2\,cos(\theta)\ cylinder]$

4. Express $\rho\, sin(\phi) = 2\, sin(\theta)$ in rectangular coordinates and sketch the graph.
Ans: [cylinder $x^2 + (y-1)^2 = 1$]

5. Express $r = 2\sec(\theta)$ in rectangular and spherical coordinates.
Ans: $[x = 2;\ \rho \sin\phi \cos\theta = 2]$

6. Express $\rho = 2\, sec(\phi)$ in rectangular and cylindrical coordinates. Ans: [z = 2]

7. Describe the surface $\rho = cos\,\phi$. Find its distinctive points. (If the surface is a paraboloid, find its vertex; if a cylinder, find its radius; if a sphere, its center and radius etc.) Ans: sphere, r = ½, c:(0,0,1/2)

8. Change $cot\,\phi = 1$ to the cylindrical and Cartesian coordinates systems.

Ans: $\left[z = r; z = \sqrt{x^2 + y^2}\right.$ inverted cone]

9. Calculate the volume of a sphere of radius R (a) by using cylindrical coordinates and (b) by using spherical coordinates. Ans: $\frac{4}{3}\pi R^3$.

10. Evaluate $\int\int\int_V z\,dx\,dy\,dz$ with $V = \{(x, y, z): 3 \le z \le \sqrt{25 - x^2 - y^2}\}$. Ans: 64π.

11. Use cylindrical coordinates to evaluate $\iiint_V 4z^3\,dx\,dy\,dz$ where V is the solid bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane z=1. Ans: $2\pi/3$

12. (a) $\int_0^{\pi/2}\int_0^2\int_{-r^2}^4 \frac{zr^3}{r\sin\theta + 4}\,dz\,dr\,d\theta$ is given in cylindrical coordinates. Rewrite the integral in rectangular coordinates. Ans: (a) $\int_0^2\int_0^{\sqrt{4-x^2}}\int_{-(x^2+y^2)}^4 \frac{z\left(x^2+y^2\right)}{y+4}\,dx\,dy\,dz$

13. Use spherical coordinates to evaluate $\iiint_V z^2\,dx\,dy\,dz$ where V is the quarter upper unit sphere centered at the origin with $y \ge 0$ and $z \ge 0$. Ans: $\pi/15$

14. $\int_0^{\pi}\int_{3\pi/4}^{\pi}\int_0^1 \rho^5\cos\theta\sin^2\phi\,d\rho\,d\phi\,d\theta$ is given in spherical coordinates. Express the integral in rectangular coordinates. Ans: $\int_{-1}^1\int_0^{\sqrt{1-x^2}}\int_{-\sqrt{1-(x^2+y^2)}}^{-\sqrt{x^2+y^2}} x(x^2+y^2+z^2)\,dz\,dy\,dx$;

15. The integral $\int_0^{\pi}\int_0^5\int_0^{\sqrt{25-r^2}} r^2\sin\theta\,dz\,dr\,d\theta$ is in cylindrical coordinates. Express it as iterated integrals in spherical coordinates. Ans: $\int_0^{\pi}\int_0^{\pi/2}\int_0^5 \rho^4\sin^3(\phi)\sin(\theta)\,d\rho\,d\phi\,d\theta$;

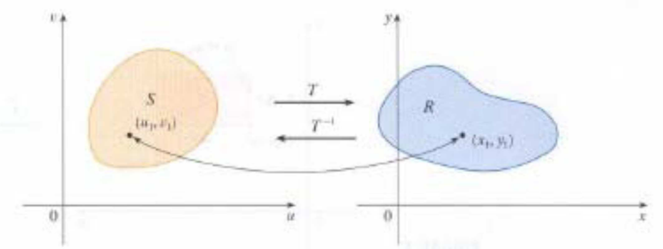
<u>**16.7 Change of Variables in a Multiple Integral (Jacobians)**</u>

Let $f = f(x)$ and $x = x(u)$. $\int_{x=a}^{x=b} f(x)dx = \int_{u=a'}^{u=b'} f(x(u))\frac{dx}{du}du$ where $\frac{dx}{du}$ is the Jacobian of the transformation in $\Re^1$.

<u>**eg 41**</u> If $y = e^{2x+1}$ and $u = 2x + 1$. $\int_{x=0}^{x=1} e^{2x+1}dx = \int_{u=1}^{u=3} e^u \frac{dx}{du}du =, \int_{u=1}^{u=3} e^u \frac{1}{2}du = \frac{e^3-e}{2}$ where $\frac{dx}{du} = \frac{1}{2}$ is the called the Jacobian of the transformation.

Transformation of Regions in $\Re^2$

A change of variables for double integrals is given by the transformation T from the u-v plane to the x-y plane where $T(u, v) = (x, y)$ where $x = g(u, v)$ and $y = h(u, v)$. Let's assume that T is a C^1 transformation, this is that g and h have continuous first partial derivative. If $T(u_i, v_i) = (x_i, y_i)$, the points (x_i, y_i) are called the image points of the points (u_i, v_i) under the transformation T. Let S be the region of all (u_i, v_i), if T transforms S into a region R in the x-y plane, R is called the image of S. If no two points in S have the same image, the transformation T is called one-to-one. If T is a one to one transformation, then it will have an inverse transformation T^{-1} that will transform points (x_i, y_i) into points (u_i, v_i), or $(u_i, v_i) = T^{-1}(x_i, y_i)$.



<u>**eg 42**</u> Draw in the x-y plane image of $S : \{(u, v)|\ 0 \le u \le 3,\ 0 \le v \le 2\}$ under the transformation

$T = \{x = 2u + 3v; y = u - v\}$.

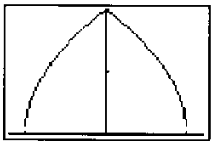
If we consider the positively oriented curve that enclose S, the line segments of the vertices of the rectangle are (0, 0), (3, 0), (3, 2), (0, 2) back to (0, 0). These four line segments in the u-v plane will map the region S to the region R in the x-y plane given by the line segments with vertices (0, 0), (6, 3), (12, 1), (6 – 1) back to (0, 0), with a *cw* orientation, under the transformation T.
Since the transformation is Affine (linear), the lines segments of the rectangle will transform into line segments.

<u>**eg 43**</u> Draw in the x-y plane the image of $S : \{(r, \theta)|0 \le r \le 2, 0 \le \theta \le \pi/2\}$ under the non-Affine transformation $T = \{x = r\,cos(\theta); y = r\,sin(\theta)\}$.

The line segments of the vertices of the rectangle of S are:
$(r : 0 \to 2,\ \theta = 0) \quad \overset{T}{\to} (x : 0 \to 2,\ y = 0)$ line segment
$(r = 2,\ \theta : 0 \to \pi/2) \overset{T}{\to} (x : 2\cos\theta\ \ y = 2\sin\theta)$ quarter circle $ccw\ \theta : 0 \to \pi/2$
$(r : 2 \to 0,\ \theta = \pi/2) \overset{T}{\to} (x = 0,\ y : 2 \to 0)$ line segment
$(r = 0;\ \theta :\ \pi/2 \to 0) \overset{T}{\to} (x = 0,\ y = 0)$ point

eg 44 Find the image of $S : \{(u, v) | \ 0 \leq u \leq 1, 0 \leq v \leq 1\}$ under the transformation the non-Affine transformation $T = \{x = u^2 - v^2; y = 2uv\}$.

The line segments of the vertices of the rectangle of S are:
$(u : 0 \to 1, v = 0) \overset{T}{\to} \ (x = u^2, \ y = 0)$ line segment $x : 0 \to 1$
$(u = 1, \ v : 0 \to 1) \overset{T}{\to} \ (x = 1 - v^2, y = 2v) \ x = 1 - \frac{y^2}{4}, (x, y) \colon (1{,}0) \to (0{,}2)$
$(u : 1 \to 0, \ v = 1) \overset{T}{\to} \ (x = u^2 - 1, y = 2u) x = \frac{y^2}{4} - 1, (x, y) \colon (0{,}2) \to (-1{,}0)$
$(u = 0; \ v : 1 \to 0) \overset{T}{\to} \ (x = -v^2, y = 0)$ line segment $x : \ -1 \to 0$



Jacobians in $\mathfrak{R}^2$

Let $x = x\,(u, v)$ and $y = y\,(u, v)$.

$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y) dxdy = \int_{v=c'}^{v=d'} \int_{u=a'}^{u=b'} f\big(x(u, v), y(u, v)\big) | \, J^{xy} \, | dudv$ where
$J^{xy} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$ is the Jacobian of the transformation in $\mathfrak{R}^2$ and $|J^{xy}|dudv$ is the element of area of the transformation. If the transformation is Affine, the Jacobian is a constant.

eg 45 Find the Jacobian in polar coordinates

Since $x = r\cos(\theta); y = rsin(\theta), J^{xy} = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -rsin\theta \\ \sin\theta & rcos\theta \end{vmatrix} = r$.
The element of area $dxdy$ in polar coordinates will be $rdrd\theta$.

Let $x = x\,(u, v, w)$; $y = y\,(u, v, w)$ and $z = z\,(u, v, w)$ $\int_{y=e}^{y=f} \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y) dxdy =$
$\int_{w=e'}^{w=f'} \int_{v=c'}^{v=d'} \int_{u=a'}^{u=b'} f\big(x(u, v, w), y(u, v, w), z(u, v, w)\big) |J^{xyz} \, | dudvdw$ where
$J^{xyz} = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} X_u & X_v & X_w \\ Y_u & Y_v & Y_w \\ Z_u & Z_v & Z_w \end{vmatrix}$ is the Jacobian of the transformation in $\mathfrak{R}^3$.

eg 46 Find the Jacobian in spherical coordinates

Since $x = \rho\, sin(\phi)\, cos(\theta); y = \rho\, sin(\phi)\, sin(\theta); z = \rho\, cos(\phi)$,

$$J^{xyz} = \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} X_\rho & X_\theta & X_\phi \\ Y_\rho & Y_\theta & Y_\phi \\ Z_\rho & Z_\theta & Z_\phi \end{vmatrix} =$$

$$\begin{vmatrix} sin(\phi)\,cos(\theta) & -\rho\, sin(\phi)\, sin(\theta) & \rho\, cos(\phi)\, cos(\theta) \\ sin(\phi)\, sin(\theta) & \rho\, sin(\phi)\, cos(\theta) & \rho\, cos(\phi)\, sin(\theta) \\ cos(\phi) & 0 & -\rho\, sin(\phi) \end{vmatrix} = -\rho^2\, sin(\phi)\ .$$

The element of area *dxdydz* in spherical coordinates will be $|J^{xyz}|d\rho d\theta d\phi = p^2\, sin(\phi) d\rho d\theta d\phi$

Integration Under Transformation in $\mathfrak{R}^2$

eg 47 Evaluate $\int\int_R \sqrt{x^2+y^2}dA$ where R is the cylinder $x^2 + y^2 = 1$ using the transformation non-affine transformation $T = \{x = rcos(\theta); y = rsin(\theta)\}$.

$\int_0^{2\pi}\int_0^1 r\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta$. Since $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} cos\,\theta & -r\, sin\,\theta \\ sin\,\theta & r\, cos\,\theta \end{vmatrix} = r,$

$\int_0^{2\pi}\int_0^1 r\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta = \int_0^{2\pi}\int_0^1 r\, rdrd\theta = \frac{2\pi}{3}.$

eg 48 Evaluate $\int\int_R\ (x+y)dxdy$ where R is the polygon with vertices (0, 0), (2, 3), (5, 1), (3, – 2) using the Affine transformation $T = \{x = 2u + 3v; y = 3u – 2v\}$.

$\int\int_R\ (x+y)dxdy = \int\int_S\ (5u+v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$ where $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -13.$

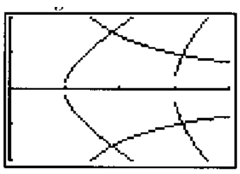
To find the limits of integration, we need to find the region S under the transformation T. To do that, it is easier if we find the inverse transformation T^{-1}.
By Cramer's Rule, $T = \{x = 2u + 3v; y = 3u – 2v\}$ becomes $T^{-1} = \{u = \frac{2x+3y}{13}; y = \frac{3x-2y}{13}\}$.

The line segments of the polygon (0, 0), (2, 3), (5, 1), (3, - 2) back to (0, 0) in the *x*-*y* are transformed to (0, 0), (1, 0), (1, 1), (0, 1) back to (0, 0) in the *u*-*v*. This is the square $S : \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$.
So the integral becomes $\int_0^1\int_0^1(5u+v)13\ dudv = 39.$

eg 49 Find $J^{uv} = \frac{\partial(u,v)}{\partial(x,y)}$ for the previous problem.

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \frac{1}{13^2}\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -\frac{1}{13}$. It can be shown that the Jacobians of Affine transformations and its inverse transformations are reciprocal of each other. So $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$.

eg 50 Evaluate $\int\int_R\ (x^2+y^2)\,cos(xy)\ dxdy$ where R is bounded by $xy = 3$, $xy = – 3$, $x^2 – y^2 = 1$ and $x^2 – y^2 = 9$.



In this case the transformation is not given, but if we use the transformation $T = \{u = xy; v = x^2 – y^2\}$, the region of integration will be the rectangle $– 3 \le u \le 3; 1 \le v \le 9$.

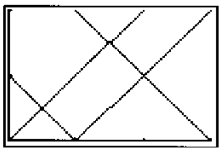
The integral $\int\int_R\ (x^2+y^2)\,cos(xy)\ dxdy = \int_1^1\int_{-3}^3(x^2+y^2)cos(u)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$

Since *x* and *y* are not expressed in terms of *u* and *v* we cannot find $\frac{\partial(x,y)}{\partial(u,v)}$ directly.

Since $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$ also holds for non-Affine transformations, we can find
$J^{uv} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = -2(x^2 + y^2)$.

The integral becomes $\int_1^1 \int_{-3}^3 (x^2 + y^2)cos(u) \left|\frac{-1}{2(x^2+y^2)}\right| dudv. = \int_1^1 \int_{-3}^3 cos(u)\frac{1}{2}dudv = 8sin(3)$.

eg 51 Evaluate $\int\int_R \left(\frac{x-y}{x+y}\right) dxdy$ where *R* is bounded by *x* – *y* = 0, *x* – *y* = 1, *x* + *y* = 1 and *x* + *y* = 3.

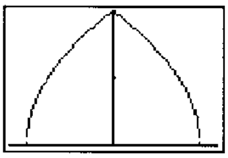


If we use the transformation *T* = {*u* = *x* – *y*; *v* = *x* + *y*}, the region of integration will be 0 ≤ *u* ≤ 1; 1 ≤ *v* ≤ 3.

The integral $\int\int_R \left(\frac{x-y}{x+y}\right) dxdy = \int_1^3 \int_0^1 \frac{u}{v} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$

Since $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}, J^{uv} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$ = 2, so the integral becomes $\int_1^3 \int_0^1 \frac{u}{v}\frac{1}{2}dudv = \frac{ln(3)}{4}$

eg 52 Evaluate $\int\int_R y\ dxdy$ where *R* is bounded by $y^2 = 4x + 4$, $y^2 = -4x + 4$, *y* ≥ 0, under the transformation *T* = {$x = u^2 - v^2$; *y* = 2*uv*}.



The condition *y* ≥ 0 implies that if *y* = 0, *u* = 0 or *v* = 0, and if *y* > 0, *uv* > 0 or *uv* < 0.

Since $y^2 = 4x + 4$ is the half parabola to the left of the region and $y^2 = -4x + 4$ is the half parabola to the right of the region, *x* and *y* into the left parabola gives

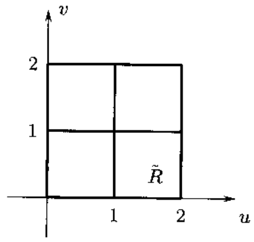
$(2uv)^2 = 4(u^2 - v^2) + 4\ or\ uv = u^2 - v^2 + 1$ with $\begin{cases} u = 0 \longrightarrow v = \pm 1 \\ v = 0 \rightarrow \text{no solution} \end{cases}$,

and *x* and *y* into the left parabola gives $(2uv)^2 = -4(u^2 - v^2) + 4$ or $uv = v^2 - u^2 + 1$ with
$\begin{cases} u = 0 \rightarrow no\ solution \\ v = 0 \rightarrow u = \pm 1 \end{cases}$ giving the region of integration 0 ≤ *u* ≤ 1; 0 ≤ *v* ≤ 1.

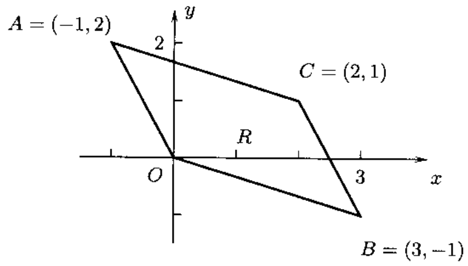
The integral then becomes $\int\int_R y\ dxdy = \int_0^1 \int_0^1 2uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \int_0^1 \int_0^1 2uv4(u^2 + v^2)dudv = 2$,
where $\frac{\partial(x,y)}{\partial(u,v)} = 4(u^2 + v^2)$.

Homework 16.7

1. Draw in an xy-plane the image of the four squares in the figure below under the affine transformation $x = 1 + 2u = v$, $y = 2 - u + 2v$.

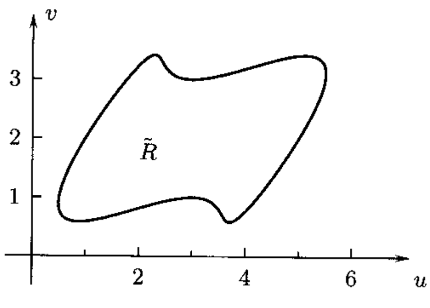


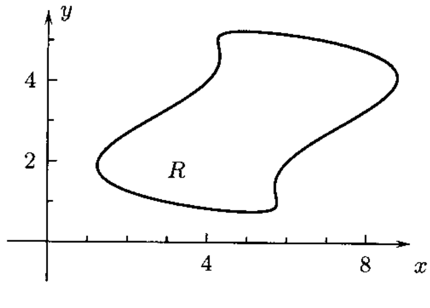
2. Find an affine transformation T : $x = a_1u + b_1v + c_1$, $y = a_2u + b_2v + c_2$ that maps the square with corners $\tilde{O} = (0, 0)$, $\tilde{A} = (1, 0)$, $\tilde{B} = (0, 1)$ and $\tilde{C} = (1, 1)$ in a uv-plane into the parallelogram in the figure below. (b) What is the Jacobian of the transformation of part (a)? (c) Find the inverse to the transformation of part (a). (d) What is the Jacobian of the inverse transformation?



Ans: $x = -u + 3v, y = 2u - v; -5; (2x + y)/5; -1/5$

3. The figures below show a region $\tilde{R}$ in a uv-plane and image R under the affine transformation $x = u + v$, $y = u - v + 2$. Use this transformation to convert $\int\int_R \frac{x-y}{x+y}\, dx + dy$ into an integral with respect to u and v over $\tilde{R}$. Ans: $\iint_{\tilde{R}} \frac{2v-2}{2u+2} du\, dv$





4. The figures below show a region $\tilde{R}$ in a *uv*-plane and its image *R* under the transformation $x = 3/u$, $y = 4v^{1/3} - 3$. Use this transformation to convert $\int\int_R x(y+3)\ dx\ dy$ into an integral with respect to *u* and *v* over $\tilde{R}$. Ans: $48\iint_{\tilde{R}} u^{-3}v^{-1/3}\, du\, dv$

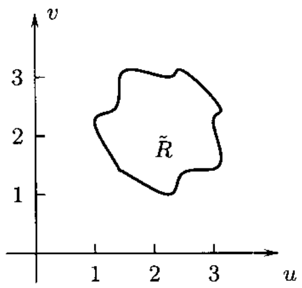


Figure 5

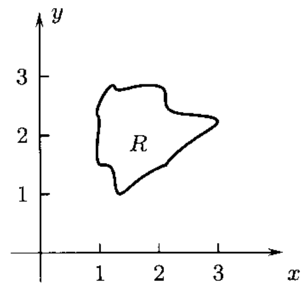
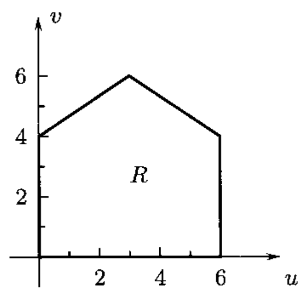


Figure 6

5. What is the Jacobian of the affine transformation $x = u + 2v + 3w$, $y = 4u - 5v$, $z = 6v + 2w$? (b) Suppose that a solid $\tilde{V}$ in *uvw*-space has volume 100 cubic meters. What is the volume of its inverse under the transformation of part (a)? 46; 4600m^3.

6. Draw in an *xy*-plane the image under the affine transformation $x = 2u + 5v$, $y = -2u + 2v$ of the pentagon in the figure below.



7. Find the values of the integrals by making affine changes of variables to obtain integrals over boxes with sides parallel to the coordinate planes: $\iiint_V \frac{x+y-z}{1+(y+2z)^2}\, dx\, dy\, dz$,
with V = {(*x*, *y*, *z*): $0 \le x + y - z \le 2$, $0 \le x - y + z \le 3$, $0 \le y + 2z \le 4$ Ans; Arctan(4)

8. Find the values of the integrals by making affine changes of variables to obtain integrals over boxes with sides parallel to the coordinate planes: $\int\int\int_V (x^2 - y^2) dx\ dy\ dz$,
with *V* = {(*x*, *y*, *z*): $1 \le x + y \le 2$, $3 \le x - y \le 4$, $4 \le x + y + z \le 5$. Ans: 21/8