Optimization: Local and Global Extrema

Functions of several variables, like functions of one variable, have local and global extrema. A local extrema is the point where the function takes on the largest or smallest value in a small region around the point. Global extrema are the largest or smallest value anywhere in the domain under consideration.

15.1 Local Extrema

2-Space

Consider the function $y = f(x)$. The critical value x_0 of the function will be found when $f'(x_0) = 0$ or $f'(x_0)$ is undefined. If the critical point $(x_0, f(x_0))$ is a local extrema, $f'(x_0) = 0$. Not all critical points are local extrema.

We can check if a critical point is a local extrema using the First Derivative Test. If the derivative changes signs across a critical point, that critical point is a local extrema.

$f(x) = x^3$ does not have a local extremum at the critical point (0,0) since the derivative does not change sign across that point.

Sometimes we can use the Second Derivative Test to find local extrema in a function.

Using this test, a critical point is a local minimum (the curve is concave up) if the second derivative is positive at that point and a local maximum (the curve is concave down) if the second derivative is negative at that point. The test fails if the second derivative is zero.

Points where the second derivative is zero are considered inflection points if concavity changes across that point. Not all points where the second derivative is zero are inflection points. $f(x) = x^3$ has an inflection point at (0,0) since the concavity changes across it. $f(x) = x^4$ has a zero second derivative at $x = 0$, but the point (0,0) is not an inflection point.

3-Space

Functions of several variables, like functions of one variable, have local and global extrema. Consider the function $z = f(x,y)$. The critical (stationary) points of the function will be found where $f_x(x,y) = f_y(x,y) = 0$. Not all critical points are local extrema.

If $F(x, y)$ has a local extremum at the point (a, b), then $f_x(a, b)$ and $f_y(a, b) = 0$.
This implies that $\vec{\nabla} f(a,b) = \vec{0}$, so the tangent plane at that point is horizontal.

eg 1 Consider $z = x^2 + y^2 - 2x - 6y + 14$.
The critical points will be found at $f_x = 2x - 2 = 0$ or $f_y = 2y - 6 = 0$. The function has a critical point at (1, 3). By completing the square, $z = 4 + (x - 1)^2 + (y - 3)^2$.

This is a paraboloid that opens up with vertex at (1, 3, 4). So this point is a local minimum.

To classify critical points we need a Second Derivative Test.

Let $z = f(x, y)$ be continuous at (a, b), $f_x(a, b) = f_y(a, b) = 0$ and lets define the Hessian determinant as $D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$.

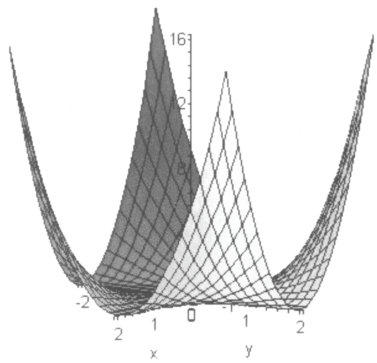
If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$ (or $f_{yy}(a, b) > 0$), then $f(a, b)$ is a local minimum.
If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$ (or $f_{yy}(a, b) < 0$), then $f(a, b)$ is a local maximum.
If $D(a, b) < 0$, then $f(a, b)$ is a saddle point.
If $D(a, b) = 0$ the test fails. In this case, further investigation is needed.

<u>eg 2a</u> Consider $z = x^4 + y^4 - 4xy + 1$.
The critical points will be found where $f_x = 4x^3 - 4y = 0$ and $f_y = 4y^3 - 4x = 0$.
Since the roots are $x = 0, 1, -1$, the critical points are $(0, 0)$, $(1, 1)$ and $(-1, -1)$.
Since $f_{xx} = 12x^2$, $f_{xy} = -4$ and $f_{yy} = 12_y{}^2$, $D(x,y) = 144x^2y^2 - 16$.
Since $D(0, 0) = -16 < 0$, $(0, 0, 1)$ is saddle point. Since $D(1, 1) = 128 > 0$ with $f_{xx}(1, 1) = 12 > 0$ and $D(-1, -1) = 128 > 0$ with $f_{xx}(-1, -1) = 12 > 0$, the points $(1, 1, -1)$ and $(-1, -1, -1)$ are local minimums.

<u>eg 2b</u>. Consider $z = x^3 + y^3 - 3x - 12y + 1$.
The critical points will be found where $f_x = 3x^2 - 3 = 0$ when $x = \pm 1$ and $f_y = 3y^2 - 12 = 0$ when $y = \pm 2$.
The critical points are $(1, 2)$, $(-1, 2)$, $(1, -2)$ and $(-1, -2)$.
Since $f_{xx} = 6x$, $f_{xy} = 0$ and $f_{yy} = 6y$, $D(x,y) = 36xy$.
Since $D(1,2) = 72 > 0$, and $f_{xx} > 0$, the point $(1,2, -17)$ is a relative minimum.
Since $D(-1, -2) = 72 > 0$, and $f_{xx} < 0$, the point $(-1, -2, 19)$ is a relative maximum.
Since $D(-1,2) = 72 < 0$, the point $(-1,2, -13)$ is a saddle point.
Since $D(1,-2) = 72 < 0$, the point $(1, -2, 15)$ is a saddle point.

<u>eg 3</u> Consider $z = x^2y^2$.

The critical points will be found where $f_x = 2xy^2 = 0$ and $f_y = 2x^2y = 0$. That is every point along the x and y axis. Since $f_{xx} = 2y^2$, $f_{xy} = 4xy$ and $f_{yy} = 2x^2$, $D(x,y) = 4x^2y^2 - 16x^2y^2 = 0$ at any point on the x or y axis. In this case the test fails. Since $x^2y^2 \geq 0$, we can conclude that we will find the minimum along the axis.



Homework 5.1

1. Classify the extreme values for f(x, y)=$4x^2e^y-2x^4-e^{4y}$
   Ans: Relative max at (-1,0) and (1, 0)

2. Find the extreme values for $f(x, y) = (y^2)/3 - (x^2)/4$
   Ans: Saddle point a (0,0)
3. Classify the two critical points (x, y, z) for $f(x, y) = 6xy - 3x^2 - 2y^3$.
   Ans: Saddle point at (0,0,0), Relative min at (1,1,1)
4. Classify the critical points of the function $f(x, y) = -x^4-y^4-4xy+1$.
   Ans: Saddle point at (0,0), relative max at (1, –1) and (–1,1)

5. Find the extreme values for $f(x, y) = x^3-3x^2y+6y^2+24y$
   Ans: Saddle points at (–2, –1) and (4,2), relative min at (0, –2)

6. Find the extreme values for $f(x, y) = x\sin y$
   Ans: Saddle points at $(0, n\pi)$

7. Find a,b,and c such that $f(x, y) = x^2+ax+y^2+by+c$ has a local minimum value of −10 at (−1,−2)
   Ans: a=2,b=4,and c=5

15.2 Optimization: Local and Global Extrema

eg 4 Find the point closest to the origin on the plane $2x + y - z = 5$.

We have minimized the square of the distance from the origin $D = x^2 + y^2 + z^2$ from the plane $2x + y - z = 5$. If we substitute z of the plane in D, we have
$D = x^2 + y^2 + (2x + y - 5)^2$. $D_x = 2x + 4(2x + y - 5) = 0$ and
$D_y = 2y + 2(2x + y - 5) = 0$ gives the system of equations $5x + 2y - 10 = 0$,
$2x + 2y - 5 = 0$. From this we get the critical point $\left(\frac{5}{3}, \frac{5}{6}\right)$. Since $D_{xx} = 10$, $D_{yy} = 4$ and $D_{xy} = 4$, $D = 10(4) - (4)^2 = 24 > 0$ and $D_{xx} = 10 > 0$, the critical point is a minimum. The z coordinate of the point is the point will be $z = 2\frac{5}{3} + \frac{5}{6} - 5 = -\frac{5}{6}$ so the closest point is $(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6})$, and the minimum distance is
$\sqrt{\left(\frac{5}{3}\right)^2 \left(\frac{5}{6}\right)^2 + \left(-\frac{5}{6}\right)^2} = \frac{5\sqrt{6}}{6}$.

Note: If we use the formula we derived with vectors, we find that the minimum distance from the point (0,0,0) to the plane $2x + y - z = 5$ is
$d = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} = \frac{|2(0)+1(0)-1(0)-5|}{\sqrt{2^2+1^2+1^2}} = \frac{|-5|}{\sqrt{6}} = \frac{5\sqrt{6}}{6}$ *units*, where <a, b, c> = <2,1,-1> is the normal to the plane.

eg 5 Find the distance from the point (2, - 3, 4) to the plane $x + 2y + 2z = 13$.

Since the distance from the point (x, y, z) to (2, - 3, 4) is given by

$d^2 = (x-2)^2 + (y+3)^2 + (z-4)^2$ and $z = \frac{13-x-2y}{2}$, into d^2 we obtain
$d^2 = (x-2)^2 + (y+3)^2 + (\frac{5-x-2y}{2})^2$. Critical point will be given where
$d_x^2 = \frac{5x}{2} + y - \frac{13}{2} = 0$ and $d_y^2 = x + 4y + 1 = 0$ this is at (3, - 1).
Since $d_{xx}^2 = \frac{5}{2}$; $d_{yy}^2 = 4$ and $d_{xy}^2 = 1$, $D = 10 - 1 = 9 > 0$, and $d_{xx}^2 > 0$, (3, - 1) is a local minimum. The minimum distance will be $d = \sqrt{(3-2)^2 + (-1+3)^2 + \left(\frac{5-3-2(-1)}{2}\right)^2} = 3$ *units*.

If we use the formula we derived with vectors, we find that the minimum distance from the point (2, - 3, 4) to the plane $x + 2y + 2z = 13$ is $d = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} = \frac{|1(2)+2(-3)+2(4)-13|}{\sqrt{1^2+2^2+2^2}} = \frac{|-9|}{3} = 3$ *units*, where $\langle a, b, c\rangle = \langle 1,2,2\rangle$ is the normal to the plane

<u>eg 6</u> The revenue obtained by selling x units of product A and y units of product B is $R(x, y) = 8x + 10y$, and the cost in producing x units of product A and y units of product B is $C(x, y) = .001(x^2 + xy + y^2) + 10{,}000$. Find the production level that maximizes profit $P(x, y)$ and the maximum profit.

Since $P(x, y) = R(x, y) - C(x, y) = 8x + 10y - .001(x^2 + xy + y^2) - 10{,}000$.
$P_x = 8 - .001(2x + y) = 0$ and $P_y = 10 - .001(x + 2y) = 0$, we obtain a critical point at $x = 2000$, $y = 4000$.
Since $P_{xx} = P_{yy} = -.002$ and $P_{xy} = -.001$. $D = 3 \times 10^{-6} > 0$. Since $P_{xx} < 0$, we have a relative maximum at that point. The maximum profit will be at the point (2000, 4000) and the maximum profit will be \$18,000.

<u>eg 7</u> An open rectangular box has volume 32 cm^3. What is the length of the edges giving the minimum surface area?

Let an open box of height z have a volume $V = xyz = 32$. The surface area will be
$S = xy + 2yz + 2xz$. If we substitute $z = 32/(xy)$, we obtain
$S = xy + 64/x + 64/y$. If we solve $S_x = y - 64/x^2 = 0$ and $S_y = x - 64/y^2 = 0$, we obtain $x = 4$, $y = 4$. Since $S_{xx} = 128/x^3$, $S_{yy} = 128/y^3$ and $S_{xy} = 1$, $D = (2)(2) - (1)^2 = 3$, and $S_{xx} = 2 > 0$, we have a minimum at (4, 4, 2). The length of the edges that gives the minimum surface area is 4 x 4 x 2 cm.

<u>eg 8</u> A rectangular box without a top is made from 12cm^2 of cardboard. Find the dimensions that maximize the volume.

If $V = xyz$, $12 = 2xz + 2yz + xy$, where xy is the area of the bottom of the box.
Since $z = \frac{12-xy}{2(x+y)}$, $V = xy\frac{12-xy}{2(x+y)} = \frac{12xy-x^2y^2}{2(x+y)}$. The critical points will be found where $V_x = \frac{y^2(12-2xy-x^2)}{2(x+y)^2} = 0$ and $V_y = \frac{x^2(12-2xy-y^2)}{2(x+y)^2} = 0$. This will occur when $x = 0$ and $y = 0$ or $12 - 2xy - x^2 = 0$ and $12 - 2xy - y^2 = 0$

The second system gives $x^2 = y^2$ or $y = x$. Since $y = x$ into either of the derivative gives $x = y = 2$, (2, 2, 1) will be the critical point. The other critical point (0, 0, 0) will give no volume. It can be shown that $D(2, 2)$ is a local maximum, so the dimensions of the box with maximum volume will be $x = 2$, $y = 2$ and $z = 1$.

Homework 5.2

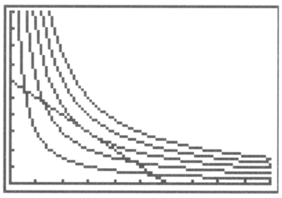
1. Find the six critical points of $g(x, y) = x^3 + y^4 - 36y^2 - 12x$.
   Ans: (2,0), (-2,0), (2, √18), (2- √18), (-2, √18), (-2, -√18)
2. Given the plane $x+2y+z-1=0$, find (a) the point on the plane closest to the origin by minimizing the distance square, (b) the minimum distance. Check you answer in (b) with the formula.
   Ans:(a) (1/6, 1/3,1/6), (b) 1/√6
3. Find the point on the half-hyperboloid $z = \sqrt{x^2 + y^2 + 3}$ that is closest to the point (6, 4, 0). (Minimize the square $f(x, y)$ of the distance between (6, 4, 0) and the point on the surface with x-coordinate x and y-coordinate y.) Ans: (3,2,4)
4. Find the point(s) (x, y, z) on the surface $3x + xy + z^2 = 9$ closest to the origin by minimizing the square of the distance. Check that the point gives a minimum. Ans: (2,1, ±1), D>0, $f_{xx} > 0$ min.
5. Find the point(s) on the cone $z^2 = x^2 + y^2$ that is closest to the point (1, 2, 0).
   Ans: (1/2, 1, ±√5/2)
6. A rectangular box of volume 24 cubic feet is to be constructed with material that costs $1.50 per square foot for the sides, $2.25 per square foot for the front and back, and $3 per square foot for the top and bottom. (a) Give a formula for the cost $C(x, y)$ of the box in terms of the width x and depth y of its base. (b) What dimensions would minimize the cost of the box?
   Ans: a) $C(x, y) = 72/x + 108/y + 6xy$. b) 2×3× 4

15.3 Constrained Optimization: Method of Lagrange Multipliers

This is a method of obtaining maximum and minimum values of a function $z = f(x, y)$ subject to a constrain $g(x, y) = k$ or $w = f(x, y, z)$ subject to a constrain $g(x, y, z) = k$ where k is a constant.

eg 9 Find the rectangle with perimeter 12 that has the largest area.

Let x and y be the length and width of the rectangle. We want to maximize $A = xy$ subject to the constrain curve $P = 2x + 2y = 12$. Graphically that means to find the point where the level curves of $A = xy$ intersecets the line (constrain curve) $2x + 2y = 12$.



At that point, the normal of the function is parallel to the normal of the constrain. So we can say $\vec{\nabla} A = \lambda \vec{\nabla} P$ where λ is a constant. So $\langle y, x \rangle = \lambda \langle 2, 2 \rangle$ or $x = \lambda 2$, $y = \lambda 2$. There are two ways to solve Lagrange Multipliers problems. We can eliminate the multiplier λ from the equations or we can solve x and y in terms of λ, substitute in the constrain to obtain an equation in λ. If we eliminate λ, we obtain $x = y$.

If we plug into the constrain $2x + 2y = 12$ we obtain $x = y = 3$ with an area $A = 9$. The point (3,3,9) is an extremum in the constrain, and we can see that any (x, y) that satisfies the extrema will give a smaller area. Therefore, the rectangle with the largest area is the square with sides 3.

In general, the method finds maximums and minimums of a function f under a constrain g, by solving the equations $\vec{\nabla} f = \lambda \vec{\nabla} g$ and the constrain.

eg 10 Find the maximum and minimum values of the plane $f(x, y) = 4x + y - 2$ under the constrain $x^2 + 2y^2 = 66$.

By the Lagrange Multipliers Method we have $4 = \lambda 2x$ and $1 = \lambda 4y$. If we substitute x and y into the constrain we obtain $\left(\frac{2}{\lambda}\right)^2 + 2\left(\frac{1}{4\lambda}\right)^2 = 66$ or $\lambda = \pm \frac{1}{4}$. For $\lambda = \pm \frac{1}{4}$ we obtain the points (8, 1) and (- 8, -1). Since $f(8, 1) = 31$ and $f(-8, -1) = -35$, (8, 1, 31) is the maximum point and (- 8, - 1 – 35) is the minimum point under the constrain.

Another way:
$4 = \lambda 2x$ and $1 = \lambda 4y$ gives that $x = 8y$ by multiplying the fist equation by 2, solving by $\lambda 4$ and equating. Into the constrain, we obtain $64y^2 + 2y = 66$ 0r $y = \pm 1$ and $x = \pm 8$.
The points will be (8, 1) and (- 8, -1) as before.

eg 11 Find the maximum and minimum values of the surface $f(x, y) = x^2 + 2x + y^2$ under the constrain $x^2 + y^2 \leq 4$.

Since we are inside a cylinder of radius 4, and the surface is not a plane, we need to check for extrema at the critical points and the boundary points.

By the Lagrange Multipliers Method, we have $2x + 2 = \lambda 2x$ and $2y = \lambda 2y$. The first equation cannot be satisfied for any value of λ. The second equation gives us $\lambda = 1$, but this value does not satisfy the first equation. The second equation also tell us $y = 0$. If we substitute $y = 0$ into $x^2 = y^2 = 4$, we obtain $x = \pm 2$. The constrain has the points (2, 0) and (- 2, 0). Since $f_x = 2x + 2 = 0$ gives $x = -1$ and $f_y = 2y = 0$ gives $y = 0$, the point (- 1, 0) is inside the cylinder and should be considered. Since $f(2, 0) = 6$, $f(-2, 0) = 2$ and $f(-1, 0) = -1$, (2, 0, 6) is the maximum point and (- 1, 0, - 1) is the minimum point.

Meaning of λ:

Consider the function $f(x, y) = x^{2/3}y^{1/3}$ under the constrain $x + y = 6$.
By the Langrange Multipliers Method, we have $\frac{2}{3}x^{-1/3}y^{1/3} = \lambda$ and $\frac{1}{3}x^{2/3}y^{-2/3} = \lambda$.
If we eliminate λ, we obtain $x = 2y$. If we substitute into the constrain, we obtain $x = 4$, $y = 2$ with $f(4, 2) = 2*2^{2/3}$ and $\lambda = 2^{2/.3}/3$

Consider the same function under the constrain $x + y = 9$.
By the Lagrange Multipliers Method, we have $\frac{2}{3}x^{1/3}y^{1/3} = \lambda$ and $\frac{1}{3}x^{2/3}y^{-2/3} = \lambda$.
If we eliminate λ, we obtain $x = 2y$. If we substitute into the constrain, we obtain $x = 6$, $y = 3$ with $f(6, 3) = 3*2^{2/3}$ and $\lambda = 2^{2/3}/3$. If we compute $\frac{\Delta f}{\Delta c}$, we obtain $\frac{\Delta f}{\Delta c} = \frac{3*2^{2/3} - 2*2^{2/3}}{9-3} = 2^{2/3}/3 = \lambda$. We can say that the change in the function by the change in the constrain is c.
Proof:
Since $x_0 = x_0(c)$ and $y_0 = y_0(c)$ both depend on the value of c, by chain rule 1,

$\frac{\partial f}{\partial c} = \frac{\partial f}{\partial x_0}\frac{dx_0}{dc} + \frac{\partial f}{\partial y_0}\frac{dy_0}{dc}$ at the critical point we have $f_x = \lambda\, g_x$ and $f_y = \lambda\, g_y$ so $\frac{\partial f}{\partial c} = \lambda\left(\frac{\partial g}{\partial x_0}\frac{dx_0}{dc} + \frac{\partial g}{\partial y_0}\frac{dy_0}{dc}\right) = \lambda\frac{\partial g}{\partial c} = \lambda$ since $g\,(x_0, y_0) = c$ and $\frac{\partial g}{\partial c} = 1$.

We can conclude that λ represents how the function changes when the value of the constrain changes.
<u>The Lagrangian Function</u>:

Constrain optimization problems where the function $f\,(x, y)$ under the constrain $g(x, y) = c$ can be solved by the Lagrangian Function
$\mathcal{L}\,(x, y, \lambda) = f\,(x, y) - \lambda\,(g(x, y) - c)$. If (x_0, y_0) is an extreme point of $f\,(x, y)$, subject to the constrain $g(x, y) = c$, and λ_0 is a corresponding Lagrange multiplier, then at the point (x_0, y_0, λ_0) we have
$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$

<u>eg 12</u> A rectangular box without a top is made from $12cm^2$ of cardboard. Find the dimensions that maximize the volume.
If $f\,(x, y, z) = xyz$ is the volume of the box, and the surface area $2xz + 2yz + xy = 12$ is the constrain $g(x, y, z) = 12$, where xy is the area of the bottom of the box, the Lagrangian function will be given by $\mathfrak{L} = xyz - \lambda\,(2xz + 2yz + xy)$.
$\frac{\partial \mathcal{L}}{\partial x} = yz - \lambda\,(2z + y) = 0$ or $xyz = \lambda\,(2xz + xy)$
$\frac{\partial \mathcal{L}}{\partial y} = xz - \lambda\,(2z + x) = 0$ or $xyz = \lambda\,(2yz + xy)$
$\frac{\partial \mathcal{L}}{\partial z} = xy - \lambda\,(2x + 2y) = 0$ or $xyz = \lambda\,(2xz + 2yz)$
$\frac{\partial \mathcal{L}}{\partial \lambda} = 2xz + 2yz + xy - 12 = 0$

if we solve the first and second equation, we have $x = y$. If we solve the first and third equation, we obtain $x = 2z$. If we substitute these two equations in the last equation, we obtain $x = y = 2$, $z = 1$.

Homework 5.3

Use Lagrange Multipliers to find the maximum and/or minimum values and the points where they occur.

1. The maximum and minimum of $f = 2x + y$ for $x^2 + 2y^2 = 18$.
   Ans: $\lambda = \pm 1/4$, max $f(4,1) = 9$, min $f(-4,-1) = -9$
2. The maximum and minimum of $f = xy$ for $4x^2 + 9y^2 = 72$.
   Ans: $\lambda \neq 0, x \neq 0, y \neq 0$, max $f(3,2) = 6$, min $f(-3,2) = f(3,-2)$ -6
3. The maximum and minimum of $f = x^2 + 2x + y^2$ for $3x^2 + 2y^2 = 48$.
   Ans: $\lambda = 0$ or $y = 0$, max $f(2 \pm \sqrt{18}) = 26$, min $f(-4,0) = 8$
4. The maximum and minimum of $f = x^2 y$ for $x^2 + 8y^2 = 24$.
   Ans: $\lambda = \pm 1$, max $f(\pm 4, 1) = 16$, min $f(\pm 4, -1) = -16$
5. The minimum of $f = 3x^2 + y^2$ for $y - 3x = 8$.
   Ans: $\lambda = 4$, min $f(-2,2) = 16$