14.1 Partial Derivatives

Let $z = f(x, y)$ be a function of two variables. If x varies while y is held fixed, z becomes a function of x, and the first partial derivative of f with respect to x will be

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

If y varies while x is held fixed, z becomes a function of y, and the first partial derivative of f with respect to y will be

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Estimating Partial Derivatives From Tables

eg 1 The wind-chill index (perceived temperature) I is a function of temperature T and wind velocity v, so we can write $I = f(T, v)$. Use the table to approximate
a) $f_T(12, 20)$; b) $f_v(12, 20)$ where T is in degree Celsius and v in km/hr.

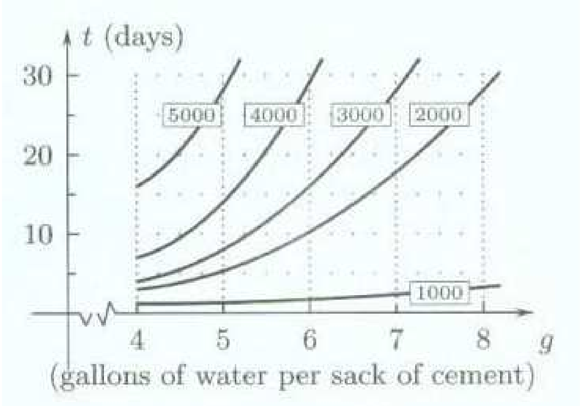
T\v	10	20	30	40
20	18	16	14	13
16	14	11	9	7
12	9	5	3	1
8	5	0	-3	-5

a) Since $f_T(12, 20) = \frac{\partial f}{\partial T} \approx \frac{f(12+\Delta T, 20) - f(12, 20)}{\Delta T}$, we can have different approximations. The left difference quotient approximation is $\frac{f(12-4,20)-f(12,20)}{-4} = \frac{0-5}{8-12} = 1.25$. The right difference quotient approximation is $\frac{f(12+4,20)-f(12,20)}{4} = \frac{11-5}{16-12} = 1.5$. The centered difference quotient approximation is $\frac{11-0}{16-8} = 1.375$.

b) Since $f_v(12, 20) = \frac{\partial f}{\partial v} \approx \frac{f(12, 20+\Delta v) - f(12, 20)}{\Delta v}$, the left difference quotient approximation is $\frac{f(12,20-10)-f(12,20)}{-10} = \frac{9-5}{10-20} = -.4$, the right difference quotient approximation is $\frac{f(12,20+10)-f(12,20)}{4} = \frac{3-5}{30-20} = -.2$ and the centered difference quotient approximation is $\frac{3-9}{30-10} = -.3$.

Estimating Partial Derivatives From Contour Diagrams

eg 2 The figure below shows the level curves of compressive strength *S*(*g*, *t*) (pounds per square inch) of Portland concrete that is made with g gallons of water per sack of cement that has cured *t* days. What are the approximate values of $\frac{\partial S}{\partial g}(6,16)$ and $\frac{\partial S}{\partial t}(6,16)$? Give the units and the everyday meaning of the answer.

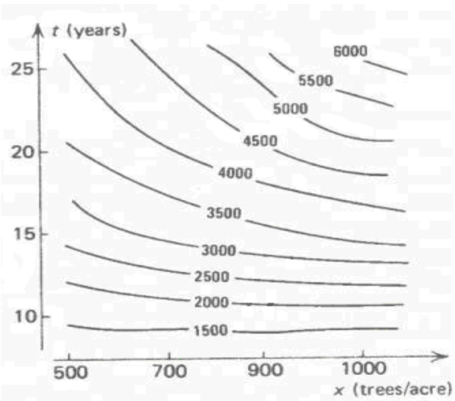


Level curves of S

$\frac{\partial S}{\partial g}(6,16) = \frac{2000-3000}{6.8-6} = -1250\ lb/sq.in/gal.$ The compressive strength decreases by 1250*lb*/*sq.in* for each gallon of water added to a sack of cement after 6 gallons of water have added.
$\frac{\partial S}{\partial t}(6,16) = \frac{4000-3000}{29-16} = 77\ lb/sq.in/day$ The compressive strength increases by 77 *lb*/*sq.in* for each day elapsed after the concrete has been cured for 16 days.

eg 3 The figure below shows the level curves of the amount of yield *Y*(*x*, *t*) (cubic *f* feet per acre) from a pine plantation of *x* trees per acre that are harvested *t* years after planted.

a) Determine without doing any calculation whether $\frac{\partial Y}{\partial t}$ (700, 20) is greater or less than $\frac{\partial Y}{\partial t}$ (1000, 15). Explain your reasoning.
$\frac{\partial Y}{\partial t}$ (700, 20) is less than $\frac{\partial Y}{\partial t}$ (1000, 15) since the level curve are further apart vertically at (700, 20) than at (1000, 15). It is better to have 1000 trees per acre than 700 trees per acre to maximize yield, because the yield increases more rapidly.



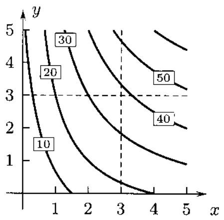
Homework 14.1

1. Use the following table of values of $g(x, y)$ to estimate (a) $g_x(2, 5)$ and (b) $g_y(2, 5)$.

	x = 1	**x = 1.5**	**x = 2**	**x = 2.5**	**x = 3**
y = 5.2	150	160	172	184	195
y = 5.0	187	200	212	223	235
y = 4.8	231	242	253	266	278
y = 4.6	273	283	293	305	316

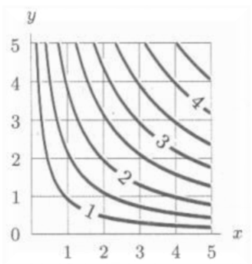
Ans: Centered {23, -202.5}

2. Level curves of $G(x, y)$ are shown in the figure below. Find its approximate x- and y-derivatives at (3, 3).



Ans: {10/1.3, 10/1.4}

3. Let the figure below be the contour diagram of $f(x,y)$. Find an approximate x derivative at (2, 2) by using the centered difference quotient. Ans:1/2



4. Figure 11 shows the graph of the function $P(x, 2)$ of x that is obtained from $P(x, y)$ by setting $y = 2$, and figure 12 shows the graph of the function $P(3, y)$ of y that is obtained from $P(x, y)$ by setting $x = 3$. Use the graphs to find the approximate values of $\frac{\partial P}{\partial x}(3, 2)$ and $\frac{\partial P}{\partial y}(3, 2)$.

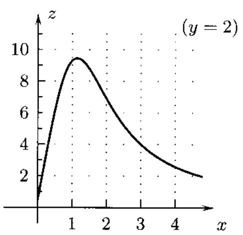


FIGURE 11

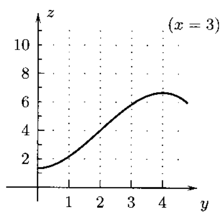
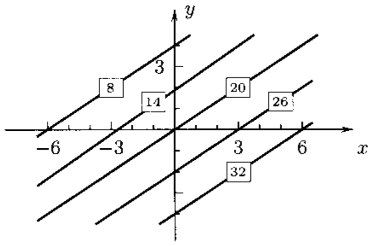


FIGURE 12

Ans: {-2, 2}

5. What are the approximate values of the x- and y-derivatives of the function $h(x, y)$ of the figure below? Find a linear equation for the level curves below. Ans: {2, -3}, $2x-3y = 20$



Level curves of $h(x, y)$

14.2 Computing Partial Derivatives Algebraically

Since $\frac{\partial f}{\partial x} = f_x(x, y)$ is the ordinary derivative of $f(x, y)$ when y is held constant and $\frac{\partial f}{\partial y} = f_y(x, y)$ is the ordinary derivative of $f(x, y)$ when x is held constant, we can use all the differentiation formulas from single variable calculus to compute partial derivatives.

eg 4 If $z = f(x, y) = x\sin(y)$, $z_x = \sin(y)$, $z_y = x\cos(y)$.

eg 5 If $z = f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$

$f_x(x,y) = 3x^2 + 2xy^3|_{(2,1)} = 16$; $f_y(x,y) = 3x^2y^2 - 4y|_{(2,1)} = 8$.

eg 6 If $z = 4x^2 - 2y + x^4y^5$, $z_x = 8x + 4x^3y^5$; $z_y = -2 + 5x^4y^4$.

eg 7 Find all first partials of $z = e^{xy}\sin(4y^2)$

$z_x = ye^{xy}\sin(4y^2)$; $\quad z_y = xe^{xy}\sin(4y^2) + 8ye^{xy}\cos(4y^2)$.

eg 8 Find a formula for the slopes of the tangent lines to a curve cut from the *elliptic paraboloid*
$z = 3x^2 + 4y^2$ parallel to the x-z plane.
$m = z_x = 6x$ for any cut with a plane $y = c$.

Partial Derivatives in Three Variables:

If $w = f(x, y, z)$, There will be three first derivative $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$ and $\frac{\partial f}{\partial z} = f_z$.

eg 9 If $w = f(x, y, z) = \sqrt{x^2 - y^2 + 2z^2}$, find the first partial derivatives.
$w_x = \frac{x}{\sqrt{x^2-y^2+2z^2}}$, $w_y = \frac{-y}{\sqrt{x^2-y^2+2z^2}}$, $w_z = \frac{2z}{\sqrt{x^2-y^2+2z^2}}$.

Implicit Partial Differentiation

eg 10 Consider the implicit function $ln\,(2x^2 + y - z^3) = x$.

By implicit partial differentiation with respect to x, $\frac{4x-3z^2z_x}{2x^2+y-z^3} = 1$ or $z_x = \frac{2x^2+y-z^3-4x}{-3z^2}$.

By implicit differentiation w.r.t. y; $\frac{1-3z^2z_y}{2x^2+y-z^3} = 0$ or $z_y = \frac{1}{3z^2}$.

eg 11 If $xy + yz = xz$, find z_x and z_y.

By implicit partial differentiation w.r.t x; $y + yz_x = z + xz_x$ or $z_x = \frac{z-y}{y-x}$.

By implicit partial differentiation w.r.t. y; $x + z + yz_y = xz_y$ or $z_y = \frac{x+z}{x-y}$.

eg 12 For $e^{xy}\, sinh(w) - z^2w + 1 = 0$, show (by using implicit partial differentiation) that

$w_x = \frac{-ye^{xy}\sin h(w)}{e^{xy}\cos h(w)-z^2}$; $w_y = \frac{-xe^{xy}\sin h(w)}{e^{xy}\cos h(w)-z^2}$; $w_z = \frac{2zw}{e^{xy}\cos h(w)-z^2}$.

Homework 14.2

1. Find (a) $\frac{\partial}{\partial x}(x^3y^2 - x + y)$, (b) $\frac{\partial}{\partial y}(x^2e^{3y} + y^2e^{3x})$, (c) $\frac{\partial}{\partial x}(xe^y + 6x^2 - y)$.

   Ans : (a) $3x^2y^2 - 1$, (b) $3x^2e^{3y} + 2ye^{3x}$, (c) $e^y + 12x$.

2. Find the x and y derivatives of (a) $F(x,y) = \sin(x^2y^4)$, (b) $G(x,y) = \ln(1-xy)$,

   (c) $H(x,y) = (x^2+x+1)(y^2+y-3)$, (d) $P(u,v) = e^{u^2}\cos(v^2)$, (e) $Q(x,y) = x^{1/2}y^{1/4} + x^2y^4$.

   Ans : (a) $F_x = 2xy^4\cos(x^2y^4)$; $F_y = 4x^2y^3\cos(x^2y^4)$, (b) $G_x = \frac{-y}{1-xy}$; $G_y = \frac{-x}{1-xy}$,

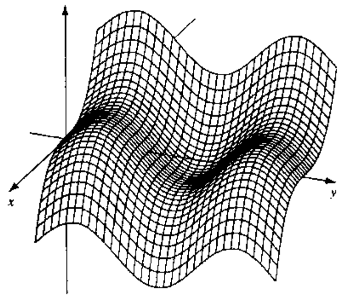
   (c) $H_x = (2x+1)(y^2+y-3)$; $H_y = (x^2+x+1)(2y+1)$, (d) $P_u = 2ue^{u^2}\cos(v^2)$; $P_v = -2ve^{u^2}\sin(v^2)$,

   (e) $Q_x = \frac{1}{2}x^{-1/2}y^{1/4} + 2xy^4$; $Q_y = \frac{1}{4}x^{1/2}y^{-3/4} + 4x^2y^3$.

3. If a gas has density ρ_0 grams per cubic centimeter at 0°C and pressure of one atmosphere, then its density at T°C and pressure P atmospheres is $\rho\,(T, P) = \frac{\rho_0 P}{1+T/273}$ gm/cm³. (a) Find formulas for $\frac{\partial\rho}{\partial T}$ and $\frac{\partial\rho}{\partial P}$ in terms of T, P, and the parameter ρ_0. Give their units. (b) One of the derivatives in part (a) is positive and the other negative for $T > -273$ and positive P. Explain in terms of gasses why this could be expected.

   Ans $\{\frac{-\rho_0 P/273}{(1+T/273)^2}, \frac{\rho_0}{1+T/273}\}$; as temperature increases, density decreases, as pressure increases density increases.

4. The figure below shows the graph of $p(x, y) = \sin y - \frac{1}{9}x^3$. (a) Find the intersection of the surface with the plane $x = 0$ and its tangent line whose slope is $\frac{\partial p}{\partial y}(0, 3\pi/4)$. (b) Find the intersection of the surface with the plane $y = 3\pi/4$ and its tangent line whose slope is $\frac{\partial p}{\partial x}(0, 3\pi/4)$.



Ans: { $p(0, y) = \sin y$, $z - \sqrt{2}/2 = -\sqrt{2}/2(y - 3\pi/4)$; $p(x, 3\pi/4) = \sqrt{2}/2 - \frac{1}{9}x^3$; $z = \sqrt{2}/2$ }

5. (a) A silo has the shape of a right circular cylinder of radius r and height H with a right circular cone of radius r and height h on top of it. Give a formula for the volume V of the silo. (b) What are the rates of change of V with respect to r, H, and h? (c) The exterior surface area of the silo is $A = 2\pi rH + \pi r\sqrt{h^2 + r^2}$. What are the rates of change of A with respect to r, H, and h?
   Ans: a) $V = \pi r^2 H + 1/3\pi r^2 h$ b) $V_r = 2\pi rH + 2/3\pi rh$; $V_H = \pi r^2$; $V_h = 1/3\pi r^2$
   c) $V_r = 2\pi H + \pi\sqrt{r^2 + h^2} + \pi r^2 / \sqrt{r^2 + h^2}$; $V_H = 2\pi r$; $V_h = \pi rh / \sqrt{r^2 + h^2}$

6. If the sides of lengths a an b in a triangle form an angle θ, use the Law of Cosines to find the third side c of the triangle. What are the rates of change of c with respect to a, b, and θ?
   Ans: $c = \sqrt{a^2 + b^2 - 2ab\cos\theta}$., $c_a = \dfrac{a - b\cos\theta}{\sqrt{a^2 + b^2 - 2ab\ \cos\ \theta}}$; $c_b = \dfrac{b - a\cos\theta}{\sqrt{a^2 + b^2 - 2ab\ \cos\ \theta}}$;
   $c_\theta = \dfrac{ab\sin\theta}{\sqrt{a^2 + b^2 - 2ab\ \cos\ \theta}}$.

7. Find z_x and z_y for:
   a) $xyz = \cos(x + y + z)$ Ans: $z_x = -\left(\frac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)}\right)$; $z_y = -\left(\frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}\right)$
   b) $x^2 + y^2 - z^2 = 2x(y+z)$ Ans: $z_x = \left(\frac{x-y-z}{x+z}\right)$; $z_y = \left(\frac{y-x}{x+z}\right)$
   c) $x^3 + y^3 + z^3 + 2xyz = 1$ Ans: $z_x = -\left(\frac{3x^2 + 2yz}{3z^2 + 2xy}\right)$; $z_y = -\left(\frac{3y^2 + 2xz}{3z^2 + 2xy}\right)$
   d) $x^2z^2 - 2xyz + y^2z^3 = 3$ Ans: $z_x = -\left(\frac{2xz^2 - 2yz}{2x^2z - 2xy + 3y^2z^2}\right)$; $z_y = -\left(\frac{-2xz + 2yz^3}{2x^2z - 2xy + 3y^2z^2}\right)$
   e) $z\sin(x\,y\,z) = -x^2$ Ans: $z_x = -\left(\frac{yz^2\cos(xyz) + 2x}{\sin(xyz) + xyz\cos(xyz)}\right)$; $z_y = -\left(\frac{xz^2\cos(xyz)}{\sin(xyz) + xyz\cos(xyz)}\right)$

8. Find z_y of $z = e^{xy}\cos(4y^2)$ Ans: $z_y = xe^{xy}\cos(4y^2) - 8ye^{xy}\sin(4y^2)$

14.3 Local Linearity and the Differential

Tangent Planes and Normal Vector:

Let $P_0 = (x_0, y_0, z_0)$ be a point on the surface $z = f(x, y)$. We know that the equation of the plane has the form $z = c + mx + ny$. Since the point $P_0 = (x_0, y_0, z_0)$ is on the tangent plane to the surface $z = f(x, y)$, $z_0 = c + m x_0 + ny_0$, or $c = z_0 - m x_0 - n y_0$. If we substitute c into the equation of the plane $z = c + mx + ny$ and rearrange, the equation of a plane tangent to the surface at the point P_0 will be given by $z = z_0 + m(x - x_0) + n(y - y_0)$, where $z_0 = f(x_0, y_0)$, $m = f_x(x_0, y_0)$ is the slope of the plane in the x direction, and $n = f_y(x_0, y_0)$ is the slope of the plane in the y direction. Since the equation of the tangent plane can be written as $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$, in vector form it becomes $< f_x(x_0, y_0), f_y(x_0, y_0), -1 > \cdot < x - x_0, y - y_0, z - z_0 > = 0$. We can see that the vector normal to the plane is $< f_x(x_0, y_0), f_y(x_0, y_0), -1 >$.

eg 13 Find the tangent plane and the normal vector to the elliptic paraboloid.
$f(x, y) = 2x^2 + y^2$ at the point (1,1).
Since $\vec{n} = < 4x, 2y, -1 > |_{(1,1,3)} = < 4, 2, -1 >$, the tangent plane will be
$z = 3 + 4(x - 1) + 2(y - 1)$ or $z = 4x + 2y - 3$, and the normal vector $< 4, 2 - 1 >$.

Local Linearization (Linear Approximation)

eg 14 Find the Local linearization for the elliptic paraboloid $z = f(x, y) = 2x^2 + y^2$ for (x, y) near the point (1,1).
If we consider the tangent plane to the surface, $z = 4x + 2y - 3$ that we obtained in the previous example, the linear function of two variables $L(x, y) = 4x + 2y - 3$ is a good approximation to the function $z = f(x, y) = 2x^2 + y^2$ when (x, y) is near the point (1, 1). This function $L(x, y)$ is called the linear approximation or the tangent plane approximation to f near (1, 1).

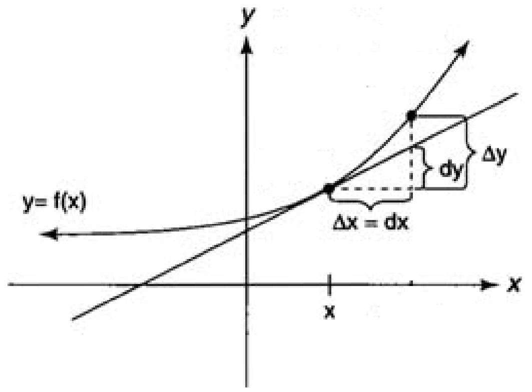
For instance $f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225$, whereas the linearization give
$L(1.1, 0.95) = 4(1.1) + 2(0.95) - 3 = 3.3$ with percentage (relative) error less than 1%.
If we take a point farther away from (1, 1), we will no longer have a good approximation. In general the linearization of (x, y) at the point (x_0, y_0) will be given by the linear approximation of the Taylor polynomial in two variables about (x_0, y_0) of $f(x, y)$ or
$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

eg 15 Find the Local linearization of $z = f(x, y) = xe^{xy}$ near the point (1, 0).

$f_x = e^{xy} + yxe^{xy}|_{(1,0)} = 1$; $f_y = x^2e^{xy}|_{(1,0)} = 1$
$L(x, y) = 1 + (x - 1) + (y - 0) = x + y$.

Differentials For Functions of Two Variables

For a function of one variable, $y = f(x)$, the differential of y is defined as
$dy = f'(x)dx$. In applications, if Δx is small, we can approximate Δy by dy.



eg 16 A circle of radius 4 is measured with an error of $\Delta r = \pm 0.01$. What is the maximum error and the percentage error in the area of the circle?

Since $A = \pi r^2$ and $\Delta r = dr$ is small, the maximum error $\Delta A \approx dA = 2\pi r\, dr =$
$2\pi\,(4)\,(\pm 0.01) = \pm 0.08\,\pi$. The relative error is $\frac{\Delta A}{A} = \frac{\pm 0.08\pi}{16\pi} = \pm 0.005$ with percentage error of $\pm 0.5\%$.

For a function to two variables, $z = f(x, y)$, the differential or total differential is defined as
$dz = f_x(x, y)\, dx + f_y(x, y)\, dy$.

eg 17 Given $z = x^2 + 3xy - y^2$
a) Find the differential dz.
b) Compare dz and Δz if x changes from 2 to 2.05 while y changes from 3 to 2.96.
a) $dz = (2x + 3y)\, dx + (3x - 2y)dy$.
b) At the point (2,3), $dz = (2(2) + 3(3))\ .05 + (3(2))\ (-.04) = 0.65$
If we compare with Δz, $\Delta z = z(2.05, 2.96) - z(2, 3) =$
$(2.05^2 + 3(2.05)\,(2.96) - 2.96^2) - (2^2 + 3(2)(3) - 3^2) = 0.6449$.
We see that dz is easier to compute, and $\Delta z \approx$ dz when Δx and Δy are small.

eg 18 The radius and height of a right–circular cone are measured with errors as much as 0.1 cm each. If the height and radius are measured to be 6 cm and 3cm respectively, use differentials to approximate the maximum possible error in the calculated value of the volume.

Since $v = \frac{1}{3}\pi r^2 h$, and $|\Delta r|$ and $|\Delta h| = 0.1$ are small, $|\Delta r| \approx dr$ and $|\Delta h| \approx dh$, so
$|\Delta v| \approx |dv| = |2/3\ \pi rh\ dr + 1/3\ \pi r^2 dh| = |2/3\ \pi\ (3)\ 6dr + 1/3\ \pi 3^2 dh| \leq$
$|1.2\pi| + |.3\pi|$ by the triangle inequality, so $|\Delta v| \leq 1.5\pi$. This will give a relative error of $\left|\frac{\Delta v}{v}\right| \leq 0.08$.

Differentials For Functions of Three Variables

Let $w = f(x, y, z)$, the differential will be $dw = f_x\,dx + f_y\,dy + f_z\,dz$.

eg 19 Estimate the largest possible error in the volume of a rectangular box that measures 75 *cm*, 60 *cm*, and 40 *cm* and each measurement within 0.2 *cm*.

Since $V = xyz$, and $|\Delta x| = |\Delta y| = |\Delta z| \le 0.2$, $|\Delta v| \approx |dv| = |v_x dx + v_y dy + v_z d_z = yzdx + xzdy + xydz|$ so $|\Delta v| \approx |dv| \le (60)(40)|(0.2)| + (75)(40)|(0.2)| + (75)(60)|(0.2)| = 1980cm^3$.
$1980cm^3$ is the maximum possible error.
This seems a large error, but if we compute the relative error, $\frac{|dv|}{v} = \frac{1980}{180000} = 0.011$, this is only 1.1% of the volume.

Homework 14.3

1. Give an equation of the tangent plane and normal vector to the graph of $z = x^2 y^{-3}$ at (2, 1).
   Ans: $z = 4(x-2) - 12(y-1) + 4$, < 4,-12,-1>.
2. Given $f(x, y) = x^2 + \ln y$, find a normal vector to the surface and the equation of the plane tangent to *f* at the point (3, 1). Ans: <6,1,-1>; 6x+y-z=10
3. Give a formula for the linear function $L(x, y)$ such that $\frac{\partial L}{\partial x} = 6$, $\frac{\partial L}{\partial y} = -3$, and $L(0, 0) = 7$.
   Ans: $L=6x-3y+7$
4. (a) Give a formula for the linear function $L(x, y)$ whose graph is the tangent plane to the graph of $g(x, y) = \sin(\pi x) + \cos(\pi y)$ at $x = 1$, $y = 1$. (b) How closely does $L(1.02, 1.03)$ approximate $g(1.02, 1.03)$?
   Ans: a) $z = -\pi(x-1) - 1$, b) 4.5×10^{-3}.
5. Given $z = x^2y + 2xy - xy^2$ find the differential dz. Ans: $dz = (2xy+2y-y^2)dx + (x^2+2x-2xy)dy$
6. The work done by a constant force of magnitude *F* on an object that moves a distance *s* along a straight line at an angle θ with the direction of the force is $W = Fs\cos\theta$. The force *F* is measured to be 10lb with an error no greater than 0.1 lb.; the distance *s* is measure to be 100ft. with an error no greater than one inch; and the angle is measured to be 30° with an error no greater that one degree. Use differentials to estimate the maximum possible error in the work if the measured values are used to calculate it. Hint: If θ is in degrees, the derivative of $\cos\theta$ becomes $\frac{d}{d\theta}\cos\left(\frac{\pi}{180}\theta\right) = -\frac{\pi}{180}\sin\left(\frac{\pi}{180}\theta\right)$. Ans: $|\Delta W| \le 18.1$ ft-lb
7. The height *h* of the frustum a right circular cone is measured to be 3 in with an error no greater than 0.01in. The radius *R* of the base and the radius *r* of the top are measured to be 10 in. and 5in with error no greater than 0.03in in each measurement. . Use differentials to estimate the maximum possible error in the computed volume $V = \frac{1}{3}\pi h(R^2 + Rr + r^2)$ of the frustum. Ans: $|\Delta V| \le 6.07$ in^3.
8. Find the tangent plane and normal vector to $z = f(x, y) = x\sin(y)$ at $(2, \frac{\pi}{2})$. Ans: z=x
9. Find the equation of the tangent plane and normal vector to $z = e^{\sin(y)}$ at $(2, \pi, 1)$. Ans: $y+z=\pi + 1$.
10. Given $z = x^2 + 3xy - y^2$, find the differential dz at the point (1,1). Ans 5dx+dy
11. Find the linearization of $f(x, y) = e^x\cos(xy)$ at $(\pi/2, 1)$. Ans: $L(x, y) = -e^{\pi/2}(x - \frac{\pi}{2}) - e^{\frac{\pi}{2}}\frac{\pi}{2}(y-1)$.

12. The radius and height of a right-circular cylinder are measured with errors of at most 0.1 inches each. If the height and radius are measured to be 10 inches and 2 inches, respectively, use differentials to approximate the maximum possible error and relative error in the calculated value of the volume.
    Ans: $|\Delta v| \approx |dv| \leq 4.4\pi$ in^3; $\left|\frac{\Delta v}{v}\right| \leq 0.11$

14.6 The Chain Rule

The **chain rule** is a formula for computing the derivative of the composition of two or more functions. That is, if f and g are functions, then the chain rule expresses the derivative of the composite function $f \circ g$ in terms of the derivatives of f and g. The counterpart to the chain rule is the substitution rule in integrals.

Chain Rule 1 (Two to One)

If $z = f(x, y)$ with $x = x(t)$ and $y = y(t)$, $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt}$.

eg 20 Let $z = (xy + 1)^2$ with $x = t^2$, $y = 2t$.

$\frac{dz}{dt} = z_x\frac{dx}{dt} + z_y\frac{dy}{dt} = 2y(xy+1)(2t) + 2x(xy+1)(2) = 4(xy+1)(yt+x)$

$= 4(2t^3 + 1)(2t^2 + t^2) = 12t^2(2t^3 + 1)$.

To check we can do the composition of functions. Since
$z = (xy + 1)^2$ with $x = t^2$, $y = 2t$, $z = (2t^3 + 1)^2$, and $\frac{dz}{dt} = 12t^2(2t^3 + 1)$.

eg 21 Let $z = f(x,y) = ln(x^2 + y^2)$, $x = 1/t, y = t^2$. Find $\frac{dz}{dt}$. Ans: $\frac{2\left(-1 + 2t^6\right)}{t\left(1 + t^6\right)}$

eg 22 Suppose the temperature at each point on a plane is given by $T = \sqrt{x^2 + y^2}$ degrees and the bug's position at time t seconds is given by $x = (t-2)^3$, $y = (t-2)^2$. Determine the rate of change of the temperature experienced by the bug as it passes through the point (1, 1). Ans : $5/\sqrt{2}$ degrees per second

eg 23 Let $z = f(x, y) = x^2y + 3xy^4$ where $x = sin(2t)$ and $y = cos(t)$. Find $\frac{dz}{dt}\big|_{t=0}$.

$\frac{dz}{dt} = (2xy + 3y^4)(2cos(2t)) + (x^2 + 12xy^3)(-sin(t))\big|_{t=0} = 6$.

eg 24 If two legs of a right triangle are increasing at 2.0 cm/min. and 3.0 cm/min. respectively, at what rates is the area increasing at the instant when both legs are 10 cm long?
Ans: 25cm^2/min

Chain Rule 2 (Two to Two)

If $z = f(x, y)$ with $x = x(s, t)$ and $y = y(s, t)$ $\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$ and $\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$.

eg 25 If $z = e^x sin(y)$ where $x = st^2$ and $y = s^2t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

$\frac{\partial z}{\partial s} = e^x sin(y)(t^2) + e^x cos(y)(2st) = e^{st^2} sin(s^2t)(t^2) + e^{st^2} cos(s^2t)(2st)$.
$\frac{\partial z}{\partial t} = e^x sin(y)(2st) + e^x cos(y)(s^2) = e^{st^2} sin(s^2t)(2st) + e^{st^2} cos(s^2t)(s^2)$.

General Chain Rule 1 in Three Variables (Three to One)

If $w = f(x, y, z)$ with $x = x(t)$, $y = y(t)$ and $z = z(t)$.

$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$

eg 26 If $w = f(x, y, z) = \sqrt{x^2 - y^2 + 2z^2}$ where $x = t^2$, $y = ln(t)$, $z = \sqrt{3 + t}$, find $\frac{df}{dt}$ at (1, 0, 2). Ans:1

General Chain rule 2 in Three Variables (Three to two)

If $w = f(x, y, z)$ with $x = x(r, s)$, $y = y(r, s)$ and $z = z(r, s)$

$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$, $\quad \frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$

General Chain rule 3 in Three Variables (Three to three)

If $w = f(x, y, z)$ with $x = x(r, s, t)$, $y = y(r, s, t)$ and $z = z(r, s, t)$

$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}, \frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$ and

$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$

eg 27 If $w = x^4y + y^2z^3$ where $x = rse^t$, $y = rs^2e^{-t}$ and $z = r^2s\,sin(t)$ find $\frac{\partial w}{\partial s}$.

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin(t)) =$

$4(rse^t)^3(rs^2e^{-t})(re^t) + ((rse^t)^4 + 2(rs^2e^{-t})(r^2s\sin(t))^3)(2rse^{-t})$

$+ (3(rs^2e^{-t})^2(r^2s\sin(t))^2)(r^2\sin(t))$.

Implicit Differentiation Formula in 2-Space

Suppose $F(x, y) = 0$ defines y implicitly as a function of x, that is if $F(x, f(x)) = 0$,
Where $x = x$, $y = f(x)$.
By chain rule 1, $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = -F_x \,/\, F_y$

eg 28 Find $\frac{dy}{dx}$ if $x^2 - xy = 0$
Since $F(x, y) = x^2 - xy$, $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(2x-y)}{(-x)} = \frac{2x-y}{x}$

eg 29 Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$.
Since $F(x, y) = x^3 + y^3 - 6xy$, $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2-6y}{3y^2-6x} = -\frac{x^2-2y}{y^2-2x}$.

Implicit Differentiation Formula in 3-Space

Suppose $F(x, y, z) = 0$ defines z implicitly as a function of x and y,
$F(x, y, f(x, y)) = 0$, where $x = x$, $y = y$, $z = f(x, y)$.
By chain rule 2, $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$. Since $\frac{dy}{dx} = 0$, $\frac{\partial z}{\partial x} = -F_x \,/\, F_z$.
$\frac{\partial z}{\partial y}$ is found in a similar way $\frac{\partial z}{\partial y} = -F_y \,/\, F_z$.

eg 30 Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ if $x^3 + y^3 + z^3 + 6xyz = 1$ by both implicit differentiation and by using formulas.

Since $F(x, y, z) = x^3 + y^3 + 6xyz - 1 = 0$,
$\frac{\partial z}{\partial x} = -F_x/F_z = -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy}$.
$\frac{\partial z}{\partial y} = -F_y/F_z = -\frac{3y^2+6xz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy}$.

Homework 14.6

1. 7. A 13-ft ladder is leaning against a vertical wall (see figure) when the man begins pulling the foot of the ladder away from the wall at a rate of 1/5 ft./s. How fast is the top of the ladder sliding down when the foot of the ladder is 5 ft. from the wall? Ans: -1/12 ft/s

2. If two resistors with resistances R_1 and R_2 are connected in parallel, the total resistance, measured in ohms (Ω), is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 and R_2 are increasing at rates 0.3 Ω/s and 0.2 Ω/s, respectively, how fast is R changing when $R_1 = 80$ Ω and $R_2 = 100$ Ω? Ans: 107/810 Ω/s.

3. Use the chain rule to find $\frac{\partial z}{\partial s}$ if $z = x^2 + y^2$; $x = st$, $y = s - t$. Check your answer by doing the composition of the functions.
   Ans: $\frac{\partial z}{\partial s} = 2st^2 - 2(s - t)$
4. Find $\frac{\partial w}{\partial s}$ if $w = \sin(2x - y)$, $x = r\sin(s)$, $y = r\,s$ by using a chain rule.
   Ans: $\frac{\partial w}{\partial s} = (2r\cos(s) - r)\cos(2r\sin(s) - r\,s)$
5. What is $\frac{d}{dt}[F(x(t), y(t))]$ at $t = 0$ if $x(0) = 3$, $y(0) = 7$, $\frac{dx}{dt}(0) = -4$, $\frac{dy}{dt}(0) = 6$, $\frac{\partial F}{\partial x}(3, 7) = 8$, and $\frac{\partial F}{\partial y}(3, 7) = 2$? Ans: -20

6. Evaluate $\frac{dy}{dx}$ for the following implicit equations:

   a) $x^2 - 2y^2 + 1 = 0$ Ans: $\frac{dy}{dx} = \frac{x}{2y}$

   b) $2\sin(xy) = 3$ Ans: $\frac{dy}{dx} = -\frac{y}{x}$

   c) $\sqrt{x^2 + 2xy + y^4} = 4$ Ans: $\frac{dy}{dx} = \frac{x+y}{2y^3 + x}$

7. Find z_x and z_y for by using the formulas derived in this section

   a) $xyz = cos(x + y + z)$ Ans: $z_x = -\left(\frac{yz+\sin(x+y+z)}{xy+\sin(x+y+z)}\right)$; $z_y = -\left(\frac{xz+\sin(x+y+z)}{xy+\sin(x+y+z)}\right)$

   b) $x^2 + y^2 - z^2 = 2x(y+z)$ Ans: $z_x = \left(\frac{x-y-z}{x+z}\right)$; $z_y = \left(\frac{y-x}{x+z}\right)$

   c) $x^3 + y^3 + z^3 + 2xyz = 1$ Ans: $z_x = -\left(\frac{3x^2+2yz}{3z^2+2xy}\right)$; $z_y = -\left(\frac{3y^2+2xz}{3z^2+2xy}\right)$

   d) $x^2z^2 - 2xyz + y^2z^3 = 3$ Ans: $z_x = -\left(\frac{2xz^2-2yz}{2x^2z-2xy+3y^2z^2}\right)$; $z_y = -\left(\frac{-2xz+2yz^3}{2x^2z-2xy+3y^2z^2}\right)$

   e) $z\,sin(x\,y\,z) = -x^2$ Ans: $z_x = -\left(\frac{yz^2\cos(xyz)+2x}{sin(xyz)+xyzcos(xyz)}\right)$; $z_y = -\left(\frac{xz^2\,cos(xyz)}{sin(xyz)+xyzcos(xyz)}\right)$

8. Use implicit differentiation to find $\frac{\partial z}{\partial y}$ if $x^2z^2 - 2xyz = 3$. Check your answer with the formulas derived in class. Ans: $\frac{z}{xz-y}$, $-\frac{F_y}{F_z} = \frac{z}{xz-y}$

9. If two sides of a right triangle are increasing at 2.0 cm/min each. At what rate is the hypotenuse of the triangle increasing at the instant when one side is 3.0 cm long and the other side is 4.0 cm long? Ans:14/5 cm/min

14.4 Gradients and Directional Derivatives

Average Rate of Change in an Arbitrary Direction:
The average rate of change of the function $f(x, y)$ at the point $P = (x_0, y_0)$ towards the point $Q = (x_0 + hu_1, y_0 + hu_2)$ in the direction of $\hat{u} = < u_1, u_2 >$ is given by

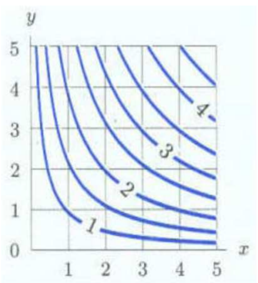
$$\frac{change\ in\ f}{Distance\ from\ P\ to\ Q} = \frac{f(x_0+hu_1, y_0+hu_2) - f(x_0, y_0)}{h}$$

The directional derivative of f at (x_0, y_0) in the direction of $\hat{u} = < u_1, u_2 >$ will be given by

$$D_{\hat{u}}f = f_{\hat{u}} = \lim_{h\to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

If $\hat{u} = \hat{\imath} =< 1, 0 >, D_{\hat{u}}f = D_{\hat{\imath}}f = \frac{\partial f}{\partial x}$. If $\hat{u} = \hat{\jmath} =< 0, 1 >, D_{\hat{u}}f = D_{\hat{\jmath}}f = \frac{\partial f}{\partial y}$.

eg 31 Estimate the directional derivative $f_{\hat{u}}(2, 3)$ in the direction of $\vec{u} = < 1, 1 >$ in the figure below:



$f_{\hat{u}}(2,3) = \frac{f(3,4)-f(2,3)}{\sqrt{2}} = \frac{3.5-2.5}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Directional Derivative and Gradient

Let $P_0\,(x_0, y_0, z_0)$ be a point on the surface $z = f\,(x, y)$, and let $p_0\,(x_0, y_0)$ be a projection of that point on the x-y plane. All points from $p_0(x_0, y_0)$ in the direction of $\hat{u} = < u_1, u_2 >$ will be given by $(x, y) = (x_0 + su_1, y_0 + su_2)$ where s is a parameter. The surface $z = f\,(x, y)$ can then be expressed as $z = f\,(x_0 + su_1, y_0 + su_2)$ where $x = x(s)$ and $y = y(s)$. Since this is a function of s, by the chain rule 1,
$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = f_x u_1 + f_y u_2$ where $\frac{dx}{ds} = u_1, \frac{dy}{ds} = u_2$.
$f_x u_1 + f_y u_2$ can be written as $< f_x, f_y > \cdot < u_1 + u_2 > = \vec{\nabla} f \cdot \hat{u}$ where
$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}$ is the *Nabla* operator and
grad f = $\vec{\nabla} f = < f_x, f_y > = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$ is the *gradient* of the function $f(x, y)$.
The directional derivative of the function $z = f\,(x, y)$ in the direction of $\hat{u}$ will be
$D_{\hat{u}}f = \frac{dz}{ds} = \vec{\nabla} f \cdot \hat{u} = < f_x, f_y > \cdot < u_1, u_2 >$.

eg 32 Find the directional derivative of $f\,(x, y) = x^2 - 3xy + 4y^3$ at $p_0 = (-2, 0)$ in the direction of $\vec{u} = < 1, 2 >$.
$D_{\hat{u}}f = \vec{\nabla} f \cdot \hat{u} = < f_x, f_y > \cdot < u_1, u_2 > =$
$< 2x - 3y, -3x + 12y^2 > |_{(-2.\,0)} \cdot < \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} > = \frac{-4}{\sqrt{5}} + \frac{12}{\sqrt{5}} = \frac{8}{\sqrt{5}}$.

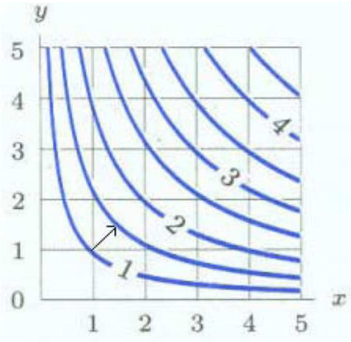
The Meaning of the Gradient of a Function

$D_{\hat{u}}f = \frac{dz}{ds} = \vec{\nabla} f \cdot \hat{u} = |\vec{\nabla} f\,||\hat{u}|\cos(\theta) = |\vec{\nabla}\, f|\cos(\theta) so\ \frac{dz}{ds} = |\vec{\nabla}\, f|\cos(\theta)$.
$\frac{dz}{ds}|_{max} = |\vec{\nabla}\, f|$when $\theta = 0,$ and $\frac{dz}{ds}|_{min} = -|\vec{\nabla}\, f|$when $\theta = \pi$.
So $|\vec{\nabla}\, f|$ gives the maximum change of the function when the vectors $\vec{\nabla} f$ and $\hat{u}$ point in the same direction, and $-|\vec{\nabla}\, f|$ gives the minimum change of the function when the vectors $\vec{\nabla} f$ and $\hat{u}$ point in opposite direction.
The gradient $\vec{\nabla} f\,(x, y)$ is a vector in $\Re^2$ pointing in the direction in which the vector increases most rapidly, and $-\vec{\nabla} f(x, y)$ is a vector in $\Re^2$ pointing in the direction in which the vector decreases most rapidly. The $\vec{\nabla} f\,(x, y)$ lies on the x-y plane below a vector tangent to the surface pointing in the direction of steepest ascent.

Gradient and Level Curves

Let $z = f\,(x, y) = k$ be the level curves of the contour diagram of the function where $x = x(t)$, $y = y(t)$, and let $\vec{r} = < x(t), y(t) >$. By chain rule 1, $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$ or $\vec{\nabla}\, f \cdot \dot{\vec{r}} = 0$, so $\vec{\nabla}\, f$ is perpendicular to $\dot{\vec{r}}$. Since $\dot{\vec{r}} = < \frac{dx}{dt}, \frac{dy}{dt} >$ is tangential to the level curves, $\vec{\nabla}\, f$ is normal to the level curves of $f\,(x, y)$.

eg 33 Draw the approximate gradient vector $\vec{\nabla} f$ (1,1) in the figure below.



The gradient will have a magnitude and a direction. Since the magnitude of the gradient is the maximum directional derivative $|\vec{\nabla} f| \approx \frac{\Delta f}{\Delta s} = \frac{1}{\sqrt{(1)^2+(1)^2}} = \frac{1}{\sqrt{2}}$, where Δs is the change in distance, and since at (1,1) direction of maximum change seems to be along the unit vector $\frac{<1,1>}{\sqrt{2}}$, the gradient vector will be (magnitude and direction) $\frac{1}{\sqrt{2}}\frac{<1,1>}{\sqrt{2}} =< \frac{1}{2}, \frac{1}{2} >$ drawn from (1, 1). (Vector shown)
Note: If the scales along the axes are not the same, the gradient vector may not look perpendicular to the level curves (contours).

eg 34 (a) Find the unit vector in the direction where $f(x, y) = 3x - ln(y)$ increases most rapidly at the point (2, 1), and (b) find the maximum rate of change of f at that point. The unit vector in the direction where f increases most rapidly is $\frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{<3,-1/y>}{\sqrt{3^2+(1/y)^2}}|_{(2,1)} = \frac{<3,-1>}{\sqrt{10}}$.
The maximum rate of change is $\frac{dz}{ds} = D_{\nabla \hat{f}}\, f = \vec{\nabla} f \cdot \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{|\vec{\nabla} f|^2}{|\vec{\nabla} f|} = |\vec{\nabla} f| = \sqrt{10}$.

eg 35 Given $f(x, y) = xe^y$ and the points P : (2,0) and Q : (0,2), find :
a) $D_{\hat{u}}\, f$ at P, in the direction from P to Q.
b) Find the direction of the maximum rate of change of f at the point P.
c) Find the maximum rate of change of the f at that point.

a) $D_{\hat{u}} f =< e^y, xe^y > \cdot \frac{<-1,1>}{\sqrt{2}}|_{(2,0)} =< 1,2 > \cdot \frac{<-1,1>}{\sqrt{2}} = \frac{1}{\sqrt{2}}$.
b) $\hat{\nabla} f = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{<1,2>}{\sqrt{5}}$
c) $D_{\hat{\nabla} f}\, f = |\vec{\nabla} f| = \sqrt{5}$.

eg 36 If $D_{\hat{u}} f\ (2,4) = \sqrt{5}$ when $\vec{u} =< -1, 2 >$ and $D_{\hat{u}} f\ (2, 4) = \sqrt{2}$ when $\vec{u} =< 1,1 >$, find $\vec{\nabla} f(2,4)$.
Since $D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u} =< f_x, f_y > \cdot < u_1, u_2 >$,
$D_{\hat{u}} f(2,4) =< a, b > \cdot < \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} > = \sqrt{5}$, and $< a, b > \cdot < \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} > = \sqrt{2}$
Where $\vec{\nabla} f = < a, b >$, will give the two equations $-a + 2b = 5$ and $a + b = 2$ with solution
$< a, b > = < \frac{-1}{3}, \frac{7}{3} > = \vec{\nabla} f$

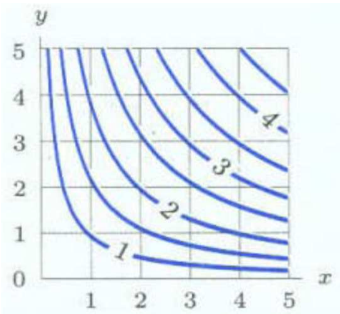
Tangent lines to contour curves.

The tangent line to a contour curve of F at the point (x_0, y_0) will be given by
$\vec{\nabla} f \cdot (\vec{r} - \vec{r_0}) = \langle f_x, f_y \rangle \cdot \langle (x - x_0), (y - y_0) \rangle = 0$ or $f_x\,(x - x_0) + f_y(y - y_0) = 0$ where $< f_x, f_y >$ is a normal vector to the contour curve at the point (x_0, y_0).

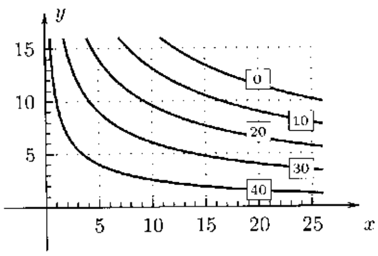
eg 37 Find the equation of the tangent line of $x^2 + y^2 = 13$ at $(2, 3)$ by viewing the curve as a contour curve of a function $f(x, y)$.
Since the function is $f(x, y) = x^2 + y^2$, the gradient is $\vec{\nabla} f(2,3) = < 2x, 2y > |_{(2,3)} = < 4, 6 >$. The equation of the line will be $4(x - 2) + 6(y - 3) = 0$.

Homework 14.4

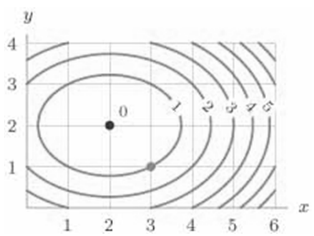
1. Estimate the directional derivative $f_{\hat{u}}(4,3)$ in the direction of $\vec{u} =< -3, 1 >$ in the figure below.
   Ans: $\frac{-3\sqrt{10}}{20}$



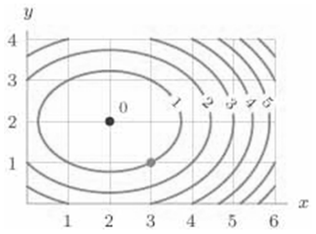
2. Level curves of a function $K(x, y)$ are shown in the figure below. Find the approximate derivative of K at the point (10, 10) in the direction towards the origin. Ans: 10/4



4. For the level curves shown below, find the approximate derivative of at (3, 1) in the direction <-2,-1>
   Ans: $2/\sqrt{5}$



5. The figure below shows level curves of $g(x, y)$. Approximate $D_{\hat{u}}g$ (3,1) with $\vec{u}$ in the direction $\vec{u} =< 3,2 >$
Ans: $5/\sqrt{13}$



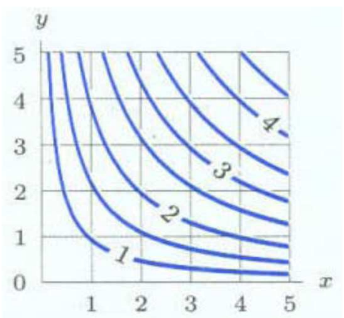
6. Find the directional derivative of $f(x, y) = xe^y$ at (1, 2) in the direction of < 3, - 4 >. Ans: $-\frac{e^2}{5}$

7. Find the directional derivative of $f(x, y) = xe^y$ at (1, 2) in the direction making an angle of $5\pi/4$ with the positive x-axis. Ans: $e^2\sqrt{2}$

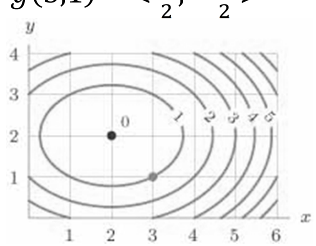
8. Given $f(x, y) = x^2 + \ln y$, find the directional derivative of f at the point (3,1) in the direction from point (3,1) to point (1,2). Ans: $-11/\sqrt{5}$

9. The temperature is given by $T(x, y) = x^3y^2$ degrees Celsius. Find (a) a unit vector in the direction in which the temperature increases most rapidly at (2, 1) and (b) the maximum rate of increase in temperature at that point. Ans: a) <3/5,4/5>; b) 20

10. Find the approximate the gradient vector $\vec{\nabla} f(4,4)$ for the function $f(x, y)$ whose level curves are shown in figure below. Ans: $\vec{\nabla} f(4,4) \approx < \frac{1}{2}, \frac{1}{2} >$



11. Find the approximate the gradient vector $\vec{\nabla} g(3,1)$ for the function $g(x, y)$ whose level curves are shown in figure below. Ans: $\vec{\nabla} g(3,1) \approx < \frac{3}{2}, -\frac{3}{2} >$



12. An ant is on a metal plate whose temperature at (x, y) is $3x^2y - y^3$ degrees Celsius. When she is at the point (5, 1), she is anxious to move in the direction in which the temperature drops the most rapidly. Give the unit vector in that direction. Ans: $-<5,12>/13$
13. An ant is on a metal plate whose temperature is $T(x, y) = 2x^2y + y^3/3$ degrees Celsius. When she is at the point (1, 1), she is anxious to move in the direction in which the temperature drops the most rapidly. Give a unit vector in that direction. Ans: $<-4,-3>/5$.
14. (a) Give the two unit vectors normal to the curve $xy^3 + 6x^2y = -7$ at (1, –1). (b)Give the two unit vectors normal to the curve $x - y^2 = 0$ at (4, 2). (c) Give the two unit vectors normal to the curve $e^{x-y2} = 1$ at (4, 2). Hint: consider each curve as a level curve of some surface $z = f(x, y)$.
    Ans: a) $\pm <-13,9>/(5\sqrt{10})$; b) $\pm <1,-4>/\sqrt{17}$; c) $\pm <1,-4>/\sqrt{17}$.
15. If $D_{\boldsymbol{u}}W(5, 10) = {}^{-}17$ for $\boldsymbol{u} = \left\langle \frac{12}{13}, -\frac{5}{13} \right\rangle$ and $D_{\boldsymbol{u}}W(5, 10) = 13\sqrt{2}$ for $\boldsymbol{u} = \langle -1,\ 1 \rangle / \sqrt{2}$. What are the x- and y-derivatives of $W(x, y)$ at (5, 10)? Ans: $W_x = -13$, $W_y = 13$
16. Find the unit vectors $\boldsymbol{u}$ such that $D_u f(x_o, y_o) = 1$ if $\vec{\nabla} f = <1, \sqrt{3}>$. Ans: $\boldsymbol{u}=\{<1,0>,<-1/2, \sqrt{3}/2>\}$
17. Find the set of points in xyz-space where $\vec{\nabla}(ye^{xz}) = <0, 1, 2>$. Ans: by inspection $z = 0,\ xy = 2$
18. Find the equation of the tangent line to the level curve $f(x, y) = \frac{x^2}{4} - \frac{y^2}{16} = 3$ at the point (4,4). Ans: $4x - y = 12$.
19. The directional derivative at $g = x^2 - 3y^3$ at (3,2) is zero. Find two vectors in those directions.
    Ans: by inspection $\pm <6,1>/\sqrt{37}$
20. Use the gradient to find the equation of the tangent line to the level curve of $f(x, y) = x^3y^2$ at the point (1,-1). Ans: 6x-2y=5

14.5 Gradients and Directional Derivatives for Functions of Three Variables

Let $w = f(x, y, z)$ be a hyper-surface in 4-space and $P_0(x_0, y_0, z_0)$ a point in 3-space. If $\hat{u} = <u_1, u_2, u_3>$ is a vector in 3-space, the directional derivative of the function $w = f(x, y, z)$ in the direction of $\hat{u}$ will be

$$D_{\hat{u}} f = \frac{dw}{ds} = \vec{\nabla} f \cdot \hat{u} = < f_x, f_y, f_z > \cdot < u_1, u_2, u_3 >$$

eg 38 Find the directional derivative of function $w = xy\sin(z)$ in the direction of $< 1, 1,1 >$.

Gradient and Level Surfaces

Let $w = f(x, y, z) = k$ be the level surface of the function where $x = x(t)$, $y = y(t)$ and $z = z(t)$. $\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$ or $\vec{\nabla} f \cdot \dot{\vec{r}} = 0$, so the gradient $\vec{\nabla} f$ is perpendicular to $\dot{\vec{r}}$, or $\vec{\nabla} f$ is normal to the level surfaces of $f(x, y, z)$ at P_0.

Tangent Planes to Level Surfaces

If $w = f(x, y, z)$ is differentiable at $P_0(x_0, y_0, z_0)$ since $\vec{\nabla} f$ is normal to the level surface, tangent plane to the level surfaces $f(x, y, z) = k$ at P_0 will be given by

$$\vec{\nabla} f \cdot (\vec{r} - \vec{r_0}) = < f_x, f_y, f_z > \cdot < (x - x_0), (y - y_0), (z - z_0) > =$$
$$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0.$$

eg 39 Find the normal unit vector and the equation of the plane tangent to the level surface $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point (2, 0, 0).

Since $f(2, 0, 0) = 2$, the level surface $\sqrt{x^2+y^2+z^2} = 2$ is a hemisphere of radius 2.
$\vec{\nabla} f = < f_x, f_y, f_z > = < \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} >.$

Since $|\vec{\nabla} f| = \frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}} = 1$, the normal $\hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}\,|_{(2,0,0)} = \hat{\imath}$.
The tangent plane will be $f_x (x-2) + f_y (y-0) + f_z (z-0)|_{(2,0,0)} = 0$, or $x - 2 = 0$.

Note: Since $z = f(x, y)$ can be considered as a level surface of the function in three variables $F(x, y, z) = f(x, y) - z = 0$, the tangent plane to F will be given by
$\vec{\nabla} F \cdot (\vec{r} - \vec{r}_0) = < f_x, f_y, -1 > \cdot < (x-x_0), (y-y_0), (z-z_0) > = 0$
$f_x(x-x_0) + f_y(y-y_0) = (z-z_0)$. This is the same result obtained for the equation of a tangent plane of a function of two variables.

eg 40 A differentiable function $f(x, y)$ has the property that $f(1, -3) = 2$, and $\vec{\nabla} f(1,-3) = < 4, 5 >$. Find (a) the equation of the tangent line to the level curve through $(1, -3)$, (b) a normal vector $\vec{N}$ to the surface at the point $(1, -3, 2)$, (c) the tangent plane to the surface $z = f(x, y)$ at $(1, -3)$.

a) The equation of the tangent line is $< f_x, f_y, > \cdot < (x - x_0), (y - y_0) > = 0$ or $4(x-1) + 5(y+3) = 0$.
b) A normal vector to the surface is $\vec{N} = < f_x, f_y, -1 > = < 4, 5, -1 >$.
c) The equation of the tangent plane is $< f_x, f_y, -1 > \cdot < (x - x_0), (y - y_0), (z - z_0) > = 0$
or $4(x-1) + 5(y+3) - (z-2) = 0$.

Homework 14.5

1. Find the directional derivative of $f(x, y, z) = x^3z - yx^2$ at $P_0(2, -1, 1)$ in the direction of $\vec{u} = < 3, -1, 2 >$.
   Ans: $\frac{34\sqrt{14}}{7}$
2. Find the equation of the tangent plane to the graph $f(x, y) = x^3y^4$ at $x = 1$, and $y = 2$
   Ans: $z = 16 + 48(x-1) + 32(y-2)$
3. Find the equation of the tangent plane to the level surface $f(x, y, z) = x^3y^2z = -4$ at $(1,2,-1)$
   Ans: $12x + 4y - 4z = 24$
4. Give the function $w(x,y,z) = x^2 + y^2 + z^2$ at the point (1,2,3),
   a) Find a normal to the level surface of w at the point (1,2,3). <2,4,6>
   b) Find the equation of the tangent plane to the level surface of w at the point (1,2,3).
   Ans: x-2y+3z=14

5. Find the equation of all normal vectors to $x = y^3z^2$ at (8,2 ,-1). Ans: $\pm$<1,−12,16>.
6. A fly is on a room whose temperature at (x, y, z) is $3x^2y - y^3 - 6z$ degrees Celsius. When she is at the point (5, 1, -20), she is anxious to move in the direction in which the temperature drops the most rapidly. Give the unit vector in that direction. Ans: $-< 5, 12, -1 >/\sqrt{170}$
7. Given $f(x, y, z) = xe^yz$ and the points $P : (2,0,1)$ and $Q : (0,2,2)$, find :
   a) $D_{\hat{u}} f$ at P, in the direction from P to Q. Ans:4/3
   b) Find the direction of the maximum rate of change of f at the point P. Ans: $\frac{<1,2,2>}{3}$
   c) Find the maximum rate of change of the f at that point. Ans: 3
8. Given $f(x,y,z) = xe^yz$, find:

a) Find the directional derivative in the direction of < 1,1,2> at the point (-1,0,1). Ans:$-2/\sqrt{6}$
b) Find a unit vector in the direction of the maximum rate of change at the point (-1,0,1).
Ans: $\frac{<1,-1,-1>}{\sqrt{3}}$
c) Find the maximum rate of change of f at the point (-1,0,1). Ans: $\sqrt{3}$

14.7 Second Order Partial Derivatives

There will be two first derivatives $\frac{\partial f}{\partial x} = f_x$ and $\frac{\partial f}{\partial y} = f_y$; four or 2^2 second derivatives $\frac{\partial^2 f}{\partial x^2} = f_{xx}$, $\frac{\partial^2 f}{\partial y^2} = f_{yy}$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$ and $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$; 2^3 third derivatives; 2^4 fourth derivatives and so on.

eg 41 If $z = f(x, y) = x\sin(y)$, $z_x = \sin(y)$, $z_y = x\cos(y)$; $z_{xx} = 0$, $z_{yy} = -x\sin y$, $z_{xy} = z_{yx} = \cos(y)$. Notice that the second mixed partials are the same since the function z is continuous.

eg 42 If $z = 4x^2 - 2y + x^4y^5$, $z_x = 8x + 4x^3y^5$; $z_y = -2 + 5x^4y^4$;
$z_{xx} = 8 + 12x^2y^5$; $z_{yy} = 20x^4y^3$; $z_{xy} = 20x^3y^4$; $z_{yx} = 20x^3y^4$.

Partial Derivatives in Three Variables:

If $w = f(x, y, z)$, There will be three first derivative $\frac{\partial f}{\partial x} = f_x, \frac{\partial f}{\partial y} = f_y$ and $\frac{\partial f}{\partial z} = f_z$; nine second derivatives $f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{xz}, f_{yz}, f_{yx}, f_{zx}, f_{zy}$, 3^3 third derivatives, 3^4 fourth derivatives and so on.

Linear and Quadratic Approximations:

The Taylor polynomial in two variables about (x_0, y_0) of $f(x, y)$ will be given by

$$f(x, y) = f(x_0, y_0) + \frac{f_x(x_0\, y_0)(x-x_0)+f_y(x_0\, y_0)(y-y_0)}{1!} + \frac{f_{xx}(x_0\, y_0)(x-x_0)^2+2f_{xy}(x_0,y_0)(x-x_0)(y-y_0)+f_{yy}(x_0,y_0)(y-y_0)^2}{2!} + \frac{f_{xxx}(x_0,y_0)(x-x_0)^3+3f_{xxy}(x_0,y_0)(x-x_0)^2(y-y_0)+3f_{yyx}(x_0,y_0)(x-x_0)(y-y_0)^2+f_{yyy}(x_0,y_0)(y-y_0)^3}{3!} + \cdots$$

The linear approximation will be
$L(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

The quadratic approximation will be
$Q(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + \frac{f_{xx}(x_0,y_0)(x-x_0)^2+2f_{xy}(x_0,y_0)(x-x_0)(y-y_0)+f_{yy}(x_0,y_0)(y-y_0)^2}{2!}$

eg 43 Find the linear and quadratic Taylor polynomial about (0, 0) of
$f(x, y) = \ln(1 + x^2 - y)$. Find $f(.5, .5)$ exact and compare with its linear and quadratic approximations
$f(0, 0) = \ln(1) = 0$, $f_x = \frac{2x}{1+x^2-y}\Big|_{(0,0)} = 0$, $f_y = \frac{-1}{1+x^2-y}\Big|_{(0,0)} = -1$,
$f_{xy} = \frac{2x}{(1+x^2-y)^2}\Big|_{(0,0)} = 0$.
Linear approximation: $L(x, y) = -y$;
Quadratic approximation: $Q(x, y) = -y + x^2 - y^2/2$;
$f(.5, .5) \approx -.28768$; $L(.5, .5) = -.5$ with 74% relative error; $Q(.5, .5) = -.375$ with 30% relative error.

Homework 14.7

1. Find all four second partial derivatives of $f(x, y) = x\cos y - y\cos x$
   Ans: $f_x = \cos y + y\sin x$; $f_y = -x\sin y - \cos x$; $f_{xx} = y\cos x$; $f_{yy} = -x\cos y$; $f_{yx} = f_{xy} = -\sin y + \sin x$

2. Find all four second partial derivatives of $f(x, y) = \tan^{-1}(y/x)$
   Ans: ; $f_{xx} = 2xy/(x^2 + y^2)^2$; $f_{yy} = -2xy/(x^2 + y^2)^2$ $f_{yx} = f_{xy} = (y^2 - x^2)/(x^2 + y^2)^2$

3. Show that the following are solutions of Laplace's equation $f_{xx} + f_{yy} = 0$.
   a) $f(x, y) = e^x\cos y$
   b) $f(x, y) = x^2 - y^2$

4. Find the linear and quadratic Taylor polynomial about $(1, -1)$ of $f(x, y) = y^2/x^3$
   Ans: $P_1 = 1 - 3(x-1) - 2(y+1)$; $P_2 = 1 - 3(x-1) - 2(y+1) + 6(x-1)^2 + 6(x-1)(y+1) + (y+1)^2$

5. Find the linear and quadratic Taylor polynomial about $(1, -2)$ of $f(x, y) = x^2y + 3y - 2$
   Ans: $P_1 = -10 - 4(x-1) + 4(y+2)$; $P_2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2)$

6. Find the quadratic (second order) Taylor polynomial about the point (0,0) for the function
   $f(x, y) = \dfrac{1}{1+2x+y^2}$. Show all the derivatives. Ans: $Q(x,y) = 1-2x+4x^2-y^2$.