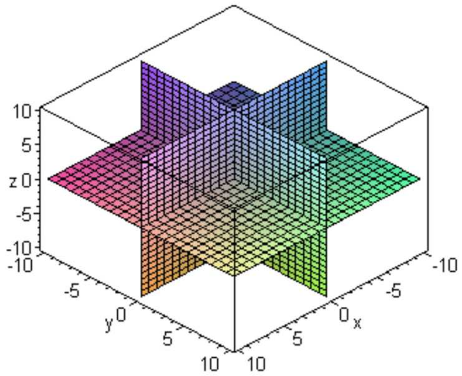
Functions of Several Variables

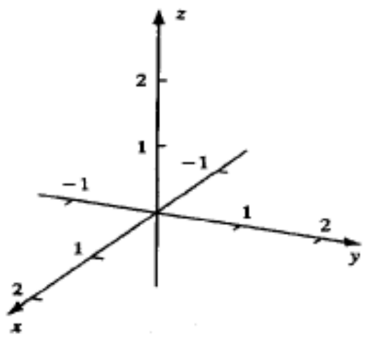
12.1 Functions of Two Variables

Rectangular Coordinate System in 3-Space

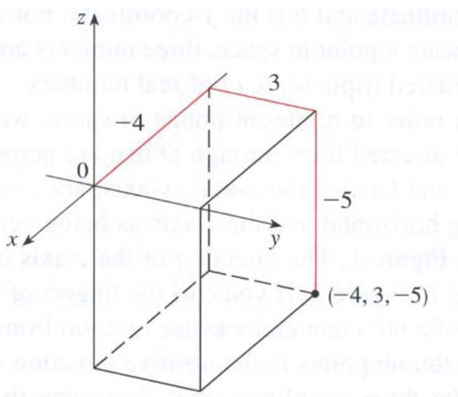
The rectangular coordinate system in $\Re^3$ is formed by 3 mutually perpendicular axes. It is a right handed system. It will define 8 octants. Each point is represented by the ordered triple (a, b, c) where a, b, c are the *x*, *y*, *z* coordinates respectively.



The first octant is formed by the positive axis.



eg 1 Sketch the point (-4, 3, -5)



A function of two variables $z = f(x, y)$ is a rule that assigns a number z to each point (x, y) in a portion of the x-y plane. The function $z = f(x, y)$ represents a surface in 3-space.

Numerical Example of a Function of Two Variables
The wind-chill index (perceived temperature) W is a function temperature T and wind velocity v, so we can write $W = f(T, v)$ where T is in degree Celsius and v in km/hr.

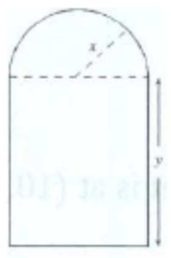
$T \backslash v$	10	20	30	40
20	18	16	14	13
16	14	11	9	7
12	9	5	3	1
8	5	0	-3	-5

eg 2. Interpret $f(12, 30) = 3$ in terms of wind-chill index.
eg 3 Interpret $f(16, v)$ in terms of wind-chill index.
eg 4 Interpret $f(T, 40)$ in terms of wind-chill index.
eg 5 If the temperature is constant, what happens to the perceived temperature as the wind velocity increases.
eg 6 If the wind is blowing at 20 km/hr, what temperature feels like 0^o C.

Algebraic Example of a Function of Two Variables

eg 7 Find a function $S(r, h)$ for the surface area of a cylinder or radius r and height h.
Answer: $S(r,h) = 2\pi rh + 2\pi r^2$

eg 8 Find a function $A(x, y)$ for the area of the Norman window shown below.



Answer: $A(x,y) = 2xy + \pi x^2 / 2$

Simple Surfaces

eg 9 What does the equation $x = 2$ represent in $\Re^2$ and $\Re^3$?

eg 10 Describe the set of points (x, y, z) such that $y = 5$ and $z = 3$.

eg 11 What does the equation $z = 0$ represents in $\Re^3$?

eg 12 Describe the region if $\Re^3$ represented by $0 \le z \le 10$.

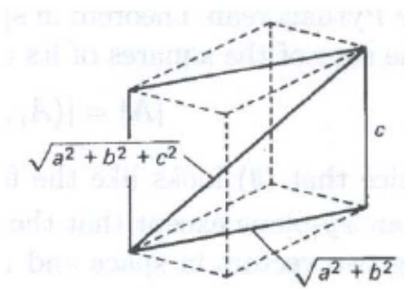
eg 13 Describe the region if $\Re^2$ and $\Re^3$ represented by $x^2 + y^2 < 1$.

eg 14 Describe the regions if $\Re^3$ represented by $x^2 + z^2 \le 4$.

Distance between two points

The distance between the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is given by

$$D = \sqrt{\left((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)\right)^2}$$



eg 15 Find the distance from the point P : (4, -5, -1) to the point Q : (3, -2, 3).

eg 16 Which of the points (-1, 3, 2), (-4, -5, 0) or (3, 2, -3) is closest to the x-z plane? Which lies on the x-y plane?

Midpoint

The midpoint of the line segment from the point (x_1, y_1, z_1) to the point (x_2, y_2, z_2) is given by

$$MP = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

eg 17 Find the midpoint of the line segment from P : (4, -5, -1) to Q : (3, -2, 3).

Spheres

The equation of a sphere with center at (x_0, y_0, z_0) and radius r is given by

$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

The general form of the equation of a sphere is given by

$x^2 + y^2 + z^2 + dx + ey + fz + g = 0$ where $d, e, f, g,$ are constants.

Not all equations of this type are spheres. If we complete the square we have $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = k$. If $k > 0$ we have a sphere of radius $r = \sqrt{k}$. If $k = 0$ we have the point (x_0, y_0, z_0). If $k < 0$ we have no real graph.

eg 18 Determine the type of graph of:
a. $x^2 + y^2 + z^2 + 10x + 4y + 2z + 28 = 0$
b. $x^2 + y^2 + z^2 + 10x + 4y + 2z + 30 = 0$
c. $x^2 + y^2 + z^2 + 10x + 4y + 2z + 32 = 0$

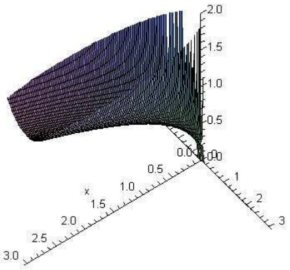
eg 19 Find the equation of the sphere with center (3, -2, 4) through (7, 2, 1).
What is the intersection with the *x*-*z* plane?

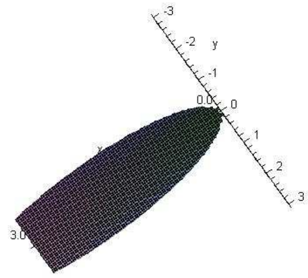
eg 20 Find the equation of the sphere with center (3, -2, 4) that touches the *x*-*z* plane.

<u>Domain of Functions of Two Variables</u>

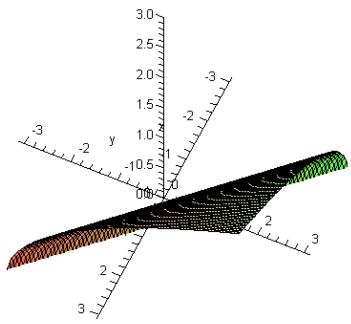
The domain of the function of two variables are the values of *x* and *y* where the function is defined. The range of the function is the set of all values of *z*.

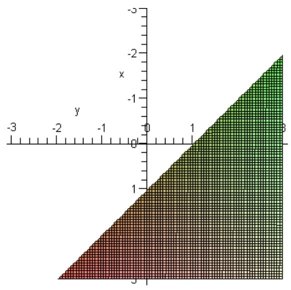
eg 21 The domain of $f(x,y) = x/\sqrt{x-y^2}$ is the inside of the sideways parabola $y^2 = x$.



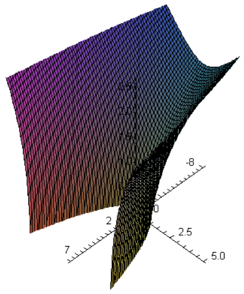


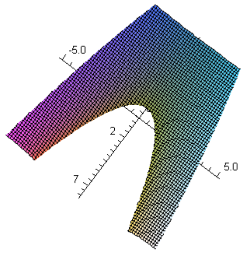
eg 22 The domain of $f(x,y) = \sqrt{x+y-1}$ is the portion of the *x*-*y* plane where
$x + y - 1 \geq 0$, $x + y \geq 1$.



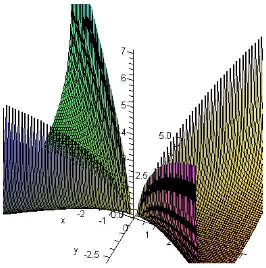


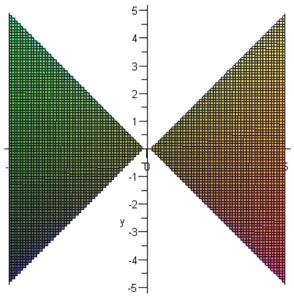
eg 23 The domain of $f(x, y) = ln\,(y^2 - x)$ is the portion of the *x*-*y* plane where $y^2 - x > 0$, or the outside of the sideways parabola $y^2 = x$.





eg 24 The domain of $f(x, y) = \dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2-y^2}}$ is the portion of the *x*-*y* plane where $x^2 - y^2 > 0 \Rightarrow (x - y)(x + y) > 0$, or the region bounded the lines $x = y$ and $x = -y$ where $(x - y)$ and $(x + y)$ are either positive or both negative.





Homework 12.1

1) Which of the points $A = (3, 0, 3)$, $B = (0, 4, 2)$, $C = (2, 4, 1)$, and $D = (2, 3, 4)$ lies closest to
   (a) the *xy*-plane?
   (b) the origin?
   (c) the *y*-axis?

2) Find which of the points A: (0,-1,2); B: (0,-4,-2) or C: (-2,2,0) is closest to the point P:(1,2,-2).

3) In words, describe the surface given by the equation $\sqrt{(x-2)^2+(y+4)^2+(z+6)^2} = 3$.

4) Sphere A is centered at the origin and the point (0, 0, 5) lies on it. Sphere B is given by the equation $x^2 + y^2 + z^2 = 5$. Are the two spheres the same?

5) What does the equation $y = 2$ represent in $\Re^2$ and $\Re^3$?

6) Describe the set of points whose distance from the *z* axis is 3.

7) Describe the set of points whose distance from the *z* axis equals the distance to the *x*-*y* plane.

8) Graph and describe the domain of $f(x,y)=\dfrac{1}{\sqrt{1-x-y}}$.

9) What is its global maximum value of $\dfrac{1}{\sqrt{1-xy}}$, and at what points does occur?

10) What is the domain of ln(xy)?

11) What is the domain of $f(x, y) = ln\,(1 - x^2 - y^2)$?

12) Find the shortest distance between the point (a,b,c) and the x-axis.

13) (a) What is the domain of $z(x, y) = \dfrac{x}{\sqrt{y}}$? (b) For what values of x and y is the function positive? For what values of x and y is z negative? For what values of x and y is z zero?

14) Find the equation tangent planes to the sphere $(x-2)^2+(y+4)^2+(z+6)^2=9$ parallel to the yz plane.

15) The table below gives the equivalent human age A(t, w) of a dog that is t years old and weighs w pounds. (a) What does A(11,50) represent and, based on the table, what is its approximate value? (b) What does A(14, 70) represent and what is its approximate value?

A(t, w) = EQUIVALENT HUMAN AGE

	t = 6	t = 8	t = 10	t = 12	t = 14	t = 16
w = 20	40	48	56	64	72	80
w = 50	42	51	60	69	78	87
w = 90	45	55	66	77	88	99

Answers 12.1 1) C,A,B; 2) C; 3) sphere of radius 3 centered at (2, -4, -6); 4) No. Sphere A is larger 5) line, plane; 6) cylinder; 7) pair of cones; 8) all points below the line x+y =1, but not the line; 9) there is none; 10) Q1 and Q3 not including the axis; 11) interior of the unit circle; 12) $d=\sqrt{b^2+c^2}$; 13) a) $y> 0$; b) $x>0, x<0, x=0$; 14) x=5, x=-1; 15) {64.5,83}.

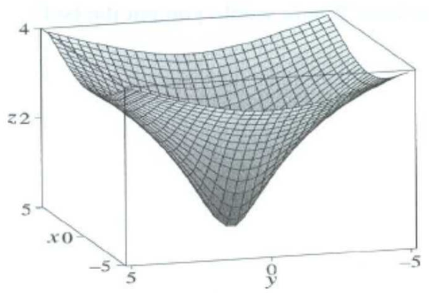
12.2 Graphs of Functions of Two Variables

Graphs of functions of two variables, $z = f(x, y)$, can be analyzed using cross-sections. These are curves obtained when the surface is sliced with planes $x = c$, $y = c$ or $z = c$. We obtain the curves by setting $x = c$, $y = c$ or $z = c$(c a constant) in the function.

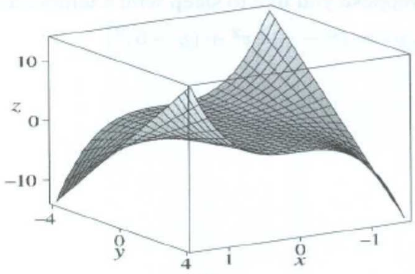
eg 25 Match each function with its graph by using cross sections. You can use an applet

1. $f(x,y) = x^3 y$ 2. $f(x,y) = \sqrt{4 - x^2 - y^2}$ 3. $f(x, y) = cos(x + y^2)$

4. $f(x, y) = ln(x^2 + y^2 + 1)$ 5. $f(x, y) = x^2 \sqrt{y}$ 6. $f(x, y) = \dfrac{1}{x+1} + siny$



I



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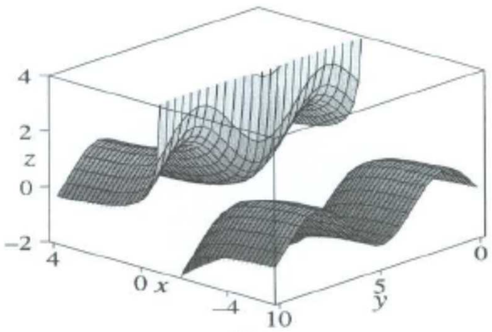
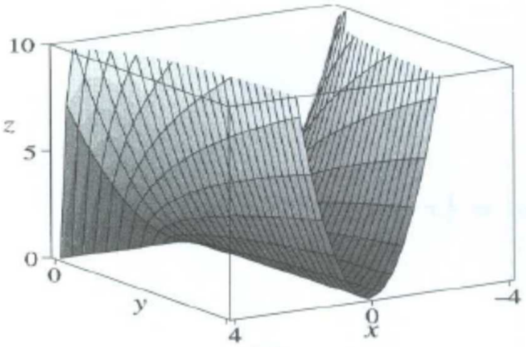


III



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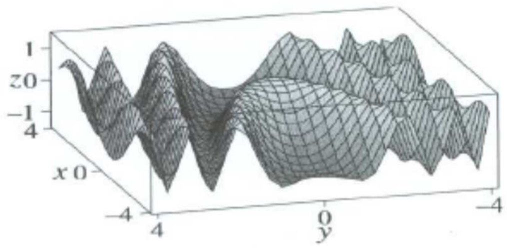

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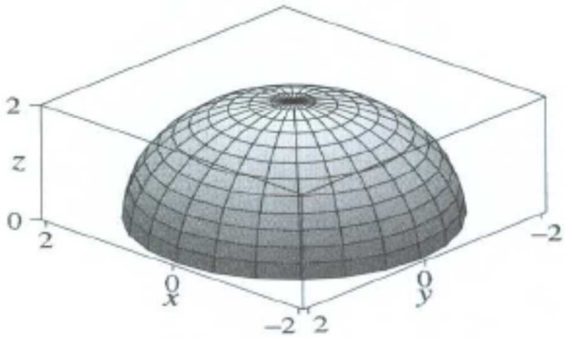


VI

Ans: 1(II), 2(VI), 3(V), 4(I), 5(IV), 6(III)

Cylindrical Surfaces (Cylinders)

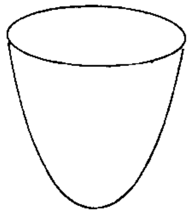
Surfaces generated when a curve on a plane is moved parallel to a line. We obtain cylindrical surfaces when one variable is missing.
eg 26 Sketch a) $y = 5$; b) $y = x^2$; c) $x^2 + z^2 = 1$; d) $z = e^{-y}$; e) $x + y = 1$;

Homework 12.2

1. (a) Is $g(x, y) = x^3y$ an increasing or decreasing function of x for $y = 10$? (b) Is $g(x, y)$ from part (a) an increasing or decreasing function of y for $x = -2$.

2. What is the global maximum of $k(x, y) = \dfrac{6}{x^2 + y^2 + 2}$ and at what point (x, y) does it occur?

3. (a) Describe the intersections of the graph $z = x^2 + y^2 + 6$ with the planes $x = c$, $y = c$, and $z = c$ for constants c. (b) Add axes to the surface in the figure below so it represents the graph of $z = x^2 + y^2 + 6$



4. Describe the level curve $N(x, y) = 1$ of $N(x, y) = \dfrac{x+2y}{3x+y}$.

5. Label positive ends of the x- and y-axes in **Figure 1** so that the surface has the equation $z = |x|$

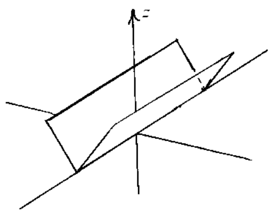


Figure 1

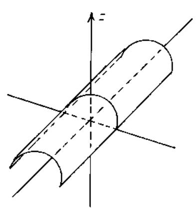


Figure 2

6. Select positive ends of the x- and y-axes in **Figure 2** so that the surface is the graph of $Q(x, y) = \sqrt{1 - y^2}$

7. Let $h(x,t) = 5 + 4\sin\left(\frac{\pi}{5}x\right)\cos(\pi t)$ be the distance above the ground (in feet) of a jump rope *x*-feet from one end after *t* seconds. If the two people turning the rope stand 5 feet apart, then $h(x,1/2)$ is flat.

8. Which of the following objects cannot be obtained as the graph of a function of two variables with a single formula?
   a) Paraboloid
   b) Plane
   c) Cylinder
   d) Sphere
   e) Line
   f) Parabolic cylinder

9. Sketch in $\Re^3$ a) $y = \sin(x)$; b) $z = \cos(y)$; c) $xy = 1$; d) $y^2 - z^2 = 1$;
   e) $\frac{x^2}{9} + \frac{z^2}{4} = 1$; f) $z = x^2 + y^2 + 1$; g) $x^2 + y^2 = 1$;

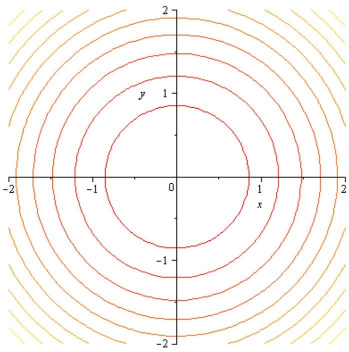
10. Describe the surfaces a) $z = 4$; b) $z = 4 - x^2 - y^2$; c) $x^2 + y^2 + z^2 = 1$.

Answers 12.2 1) a) inc. b) dec.; 2) 3; (0,0); 3) $x = c$ and $y = c$ are parabolas that open to the *z*-axis, $z=c$ is a circle for $z > 6$; 4) line $y = 2x$; $(x, y) \neq (0,0)$; 7) True; 8) (c,d,e); 10) a) plane, b) paraboloid c) sphere.

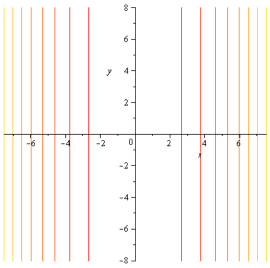
12.3 Contour Diagrams

Functions of two variables can be represented by graphs and by contour curves.
The *level (contour) curves* of the function of two variables, $z = f(x, y)$, are curves obtained by slicing a surface with horizontal planes. Algebraically, we can find them where the function is constant or $z = f(x, y) = k$; k a constant.

eg 27 The *level curves* of the paraboloid that opens downward with vertex at (0, 0, 4) $z = 4 - x^2 - y^2$ are concentric circles with center at (0, 0) including (0, 0).



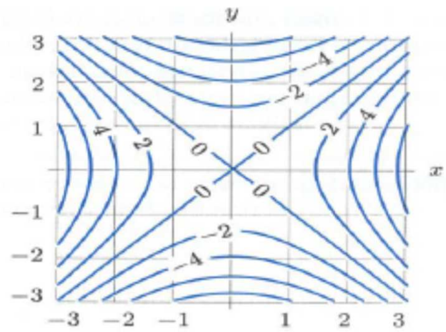
eg 28 (a) Describe the level curves of $f(x, y) = x^2$. The contour map of the function is given below.



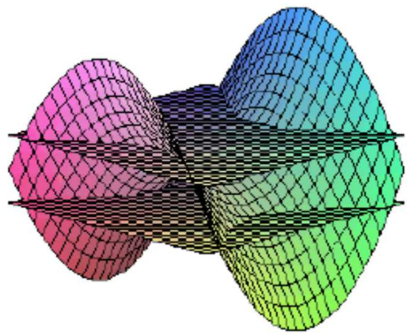
$f(x, y) = x^2$ is a parabolic cylinder aligned with the y-axis, and the contour lines are parallel lines. The closer the lines, the steeper the surface.

eg 28 (b) Describe the *level curves* of $f(x, y) = x^2 - y^2$. The contour map and the table of values of the function are given below.

y \ x	−3	−2	−1	0	1	2	3
3	0	−5	−8	−9	−8	−5	0
2	5	0	−3	−4	−3	0	5
1	8	3	0	−1	0	3	8
0	9	4	1	0	1	4	9
−1	8	3	0	−1	0	3	8
−2	5	0	−3	−4	−3	0	5
−3	0	−5	−8	−9	−8	−5	0

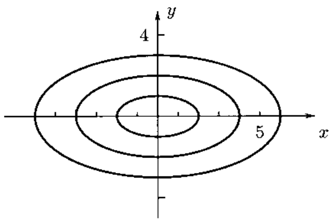


The curves are hyperbolas aligned with the x-axis ($z > 0$), the y-axis ($z < 0$) or the origin ($z = 0$).

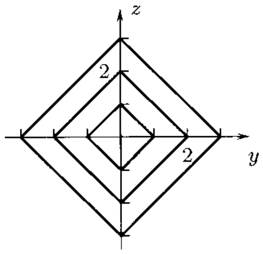


Homework 12.3

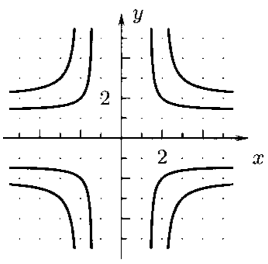
1. What are the values of $P(x, y) = x^2 + 4y^2$ on its three level curves in the figure below?



2. What are the values of $L(x, y) = |x| + |y|$ on its three level curves in the figure below?



3. What are the values of $K(x, y) = \dfrac{1}{x^2} + \dfrac{1}{y^2}$ on its eight level curves in the figure below?



4. Draw (a) the graph of R $(x,y) = \sqrt{4 - x^2}$. (b) Describe the level curves where the function of part (a) has the value 0, 1, and 2.

5. Which of the following is true?
   a) The values of contour lines are always 1 unit apart.
   b) Any contour diagram that consists of parallel lines comes from a plane.
   c) The contour diagram of any plane consists of parallel lines.
   d) Contour lines can never cross.
   e) The closer the contours, the steeper the graph of the function.

Answers 12.3 1) $P_{in} = 4$; $P_{mid} = 16$; $P_{out} = 36$; 2) $L_{in} = 1$; $L_{mid} = 2$; $L_{out} = 3$; 3) all $K_{in} = 1/2$; all $K_{out} = 2/9$: 4) Lines $x = \pm 2,\ x = \pm\sqrt{3},\ x = 0$; 5) (c,e)

12.4 Linear Functions

A linear function in $\Re^3$ will have the form $f(x, y) = c + mx + ny$, where m is the slope in the x direction, n is the slope in the y direction and c is the z intercept. The graph of a linear function in $\Re^3$ is the graph of a plane.
If the plane passes through the point (x_0, y_0, z_0), the equation of the plane becomes
$f(x, y) = z_0 + m(x - x_0) + n(y - y_0)$.

eg 29 (slope-intercept): Find the equation of the linear function with slope 0 in the direction of the x-axis, slope – 2 in the direction of the y-axis and cuts the z-axis at $z = 4$.
Sketch the function. ANS: $z = 4 - 2y$

eg 30 (point-slope): Find the equation of the linear function with slope – 4 in the direction of the x-axis, slope – 2 in the direction of the y-axis through the point (1, 2, – 4).
Sketch the function. ANS: $z = 4 - 4x - 2y$

eg 31 Find the equation of the linear function of the form $z = c + mx + ny$ whose graph intersects the x-y plane in the line $x = -3y - 2$, and intersects the x-z plane in the line $x = -5z - 2$. ANS: $z = -2/5 - x/5 + 3y/5$

eg 32 (point-slope) Find the equation of the linear function through the points (1, 2, – 1), (2, 2, 1), (1, –1, 2)
The points with the same x-coordinates will give the slope in the direction of y; or $\frac{\Delta z}{\Delta y} = \frac{-3}{3} = -1$.
The points with the same y-coordinates will give the slope in the direction of x; or $\frac{\Delta z}{\Delta x} = \frac{2}{1} = 2$.
So $z = c + 2x - y$. We can find c with any of the given points, so $z = -1 + 2x - y$.

eg 33 Find the equation of the linear function in the table below.

$x\backslash y$	1	3	4	5
1	4	-4	-8	-12
2	7	-1	-5	-9
3	10	2	-2	-6
4	13	5	1	-3

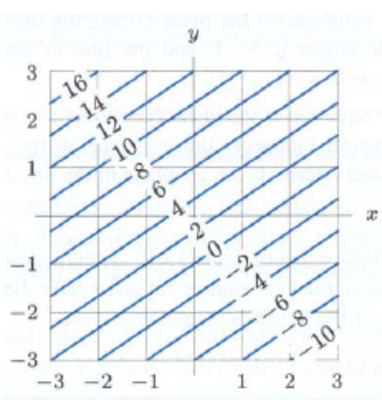
ANS; $z = 3x - 4y + 5$

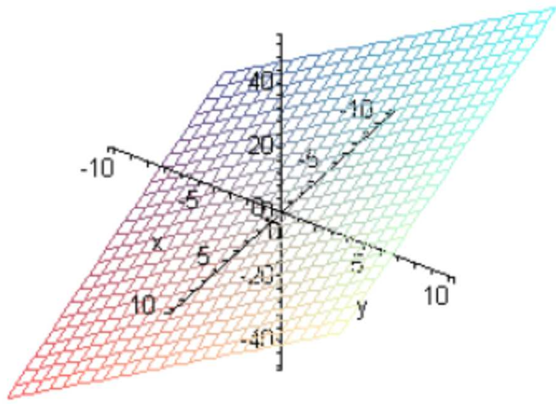
eg 34 Draw the plane 2x+3y+6z=12 in the first octant.

Contour Maps of Linear Functions

Contour maps of linear functions are parallel lines with equally spaced contour lines.

eg 35 Find a linear function for the contour map below

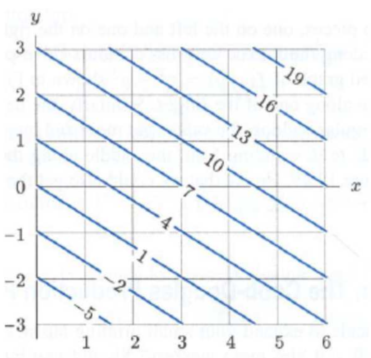




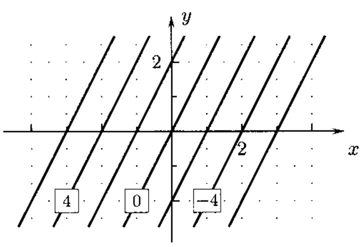
Starting at (0, 1), one unit in x, decreases z from 6 to 4, so $\Delta z/\Delta x = -2$. Starting at (0, 1), two units in y, increases z from 6 to 12, so $\Delta z/\Delta y = 3$. Since the plane passes through (0, 0, 3) the equation of the plane is $z = 3 - 2x + 3y$.

Homework 12.4

1) Find a linear function for the contour map below.



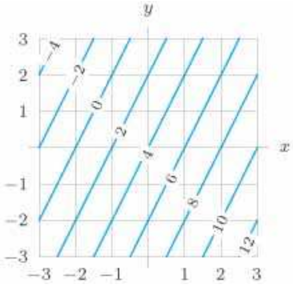
2) The figure below shows level curves of the function $F(x, y) = Ax + By + C$. What are the values of the constants A, B, C?



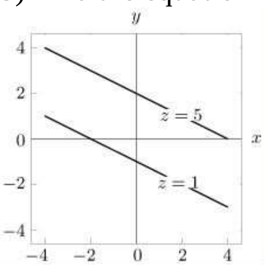
3) Find equations for linear functions with the given values.

$x\backslash y$	-1	0	1	2
0	1.5	1	0.5	0
1	3.5	3	2.5	2
2	5.5	5	4.5	4
3	7.5	7	6.5	6

4) Find a linear function for the contour map below.



5) Find the equation $z=f(x,y)$ for the linear of the partial contour diagram below.



6) Which of the following planes are parallel?

$(a)\ \ z=-2-2x-4y$

$(b)\ \ z=-1-x-2y$

$(c)\ \ (z-1)=-2-2(x-1)-4(y-1)$

$(d)\ \ z=2+2x+4y$

7) True/False: Any three points in 3-space determine a unique plane. Explain.

8) Find the function linear (equation of the plane) through intercepts (0,0,2), (0, –3,0) and (4,0,0).

9) Find the equation of the linear function z = m*x*+n*y*+c whose graph intersects the *yz*-plane in the line *z*=3*y*+5 and the *xz*-plane in the line *z*= –2*x*+5.

Answers 12.4 1) *f*(*x*, *y*) = 2x + 3y + 1; 2) –2,1,0; 3) *z*=1+2*x*-*y*/2; 4) z=2x–y+4; 5) z=7/3+2x/3+4y/3; 6) (a,c); 7) False: points cannot be collinear; 8) 3x–4y+6z=12; 9) *z*= –2*x*+3*y*+5.

12.5 Functions of Three Variables

A functions of three variables is a rule that assigns to each ordered triple (x, y, z) a unique value $w = f(x, y, z)$. These functions are difficult to visualize since they are in 4-space. The temperature T at a point at an instant of time, can be determined by $T = T(x, y, z)$ for every point in space.

Domain in Three Variables

The domain of the function of three variables are the values of $x, y,$ and z where the function is defined. The range of the function is the set of all values of $w = f(x, y, z)$.

eg 36 Find the domain of w = $\sqrt{z - x^2 - y^2}$.
Since $z - x^2 - y^2 \geq 0$, or $z \geq x^2 + y^2$, the domain is the inside of the paraboloid $z = x^2 + y^2$ including the surface.

eg 37 Find the domain of $w = \dfrac{1}{x^2 + y^2 + z^2 - 1}$.
The domain is inside and outside of the sphere of radius 1 but not the unit sphere.

Quadric Surfaces

A quadric surface is the graph of a second order degree in three variables x, y, z.
The most general form of the quadrics is
$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$.
If we neglect rotations we get the form
$Ax^2 + By^2 + Cz^2 + Gx + Hy + Iz + J = 0$.
If we neglect translations and rotations we get the form
$Ax^2 + By^2 + Cz^2 + J = 0$.
These are the counter part of the conic sections in the plane.
Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; cones $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$; paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$
Hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$; Hyperboloid one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
Hyperboloid two sheets $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

To analyze the quadrics we can use the traces (conic curves obtained when one of the variables in a quadrics is replaced by a constant).

eg 38 The traces of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ are ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ when $z = 0$; ellipses $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$ when $y = 0$; ellipses $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ when $x = 0$.
eg 39 The traces of the Hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ are ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ when $z = 0$; hyperbolas $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ when $y = 0$; hyperbolas $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ when $x = 0$.

Level Surfaces

We can study functions of three variables by examining its *level surfaces*, which are surfaces obtained when $c = k$ a constant.

eg 40 $f(x, y, z) = x^2 + y^2 - z^2 = k$, is a pair of cones if $k = 0$, a Hyperboloid of one sheet if $k > 0$ and a hyperboloid of 2 sheets if $k < 0$.

Homework 12.5

1. Sketch the surface $x^2 + y^2 - z^2 = K$ for $K = -1$ and $K = 1$.
2. Sketch the surface $1 = x^2 - y^2 - z^2$.
3. Sketch $z^2 = 4x^2 + y^2 + 8x - 2y + 4z$.
4. Identify the surface and find the center of $x^2 - y^2 + z^2 = -6x + 6y - 4z$.
5. For the surface $36z = 4x^2 + 9y^2$ find the focus and the vertex of the parabolic trace in the plane $x = 3$.
6. Let $f(x, y, z) = y^2 - x^2$. Sketch the level surface that passes through the point $(1, \sqrt{2}, 1)$.
7. Level surfaces of the function $f(x, y, z) = (x^2 + y^2)^{-1/2}$ are:
   (a) Circles centered at the origin.
   (b) Spheres centered at the origin.
   (c) Cylinders centered around the z-axis.
   (d) Upper halves of spheres centered at the origin.

8. Describe in words the level surfaces (with its alignment) of $f(x, y, z) = x^2 + z^2$.
9. Describe the level surfaces of $V(x, y, z) = \ln(x^2 + y^2 + z^2)$.
10. Can the level surface $x^2 + y^2 + z^2 = 1$ be expressed as a function z=f(x,y)? Explain
11. True or false? If false, explain.
    (a) Any level surface of a function of 3 variables can be thought of as a surface in 3-space.
    (b) Any surface that is a graph of a 2-variable function $z = f(x, y)$ can be thought of as a level surface of a function of 3 variables.
    (c) Any level surface of a function of 3 variables can be thought of as the graph of a function $z = f(x, y)$.
12. Describe in words the level surfaces of $F(x, y, z) = x^2 + z^2 + 1$

Answers 12.5 1) [hyp two sheets; hyp one sheet] both aligned with the z-axis (axis of symmetry); 2) hyp two sheets aligned with the x-axis; 3) [hyp one sheet c: (-1, 1, 2)]; 4) [hyp one sheet c(-3, -3, -2)]; 5) [f:(3, 0, 2), v:(3, 0, 1)]; 6) A cylindrical surface of a unit hyperbola along the z-axis; 7) c; 8) cylinders aligned with the y-axis of radius $\sqrt{K}$; 9) If V=b >0, spheres of radius $e^{b/2}$ with center at the origin; 10) No; 11) (F,T,F); 12) F=b>1, concentric cylinders aligned with the y-axis.

12.6 Limits and Continuity

Limits

The function $f(x, y)$ has a limit L at the point (a, b), ($\lim_{(x,y)\to(a,b)} f(x,y) = L$) if $f(x,y)$ gets close to L when the distance from the point (x, y) to the point (a, b) is sufficiently small, but not zero.

eg 41 $\lim_{(x,y)\to(\frac{1}{2},\pi)} xy^2 \sin(xy) = \frac{\pi^2}{2}$

eg 42 $\lim_{(x,y)\to(0,0)} \frac{1}{x^2+2y^2} = \infty$, so the limit does not exist.

eg 43 $\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)} = \lim_{r\to0^+} \frac{1-\cos(r^2)}{(r^2)} = \lim_{r\to0^+} \frac{2r\sin(r^2)}{2r} = 0$ where a change to polar coordinates was made, and L'Hospital rule was applied.

eg 44 $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)} = \lim_{r\to0^+} \frac{\sin(r^2)}{r^2} = 1$ where a change to polar coordinates was made, and L'Hospital's rule was applied.

eg 45 $\lim_{(x,y)\to(0,0)} e^{-\frac{1}{x^2+y^2}} = \lim_{r\to0^+} = \frac{1}{e^{\frac{1}{r^2}}} = 0$ where a change to polar coordinates was made, and L'Hospital rule was applied.

eg 46 $\lim_{(x,y)\to(0,0)} y\ln(x^2 + y^2) = \lim_{r\to0^+} r\sin(\theta) \ln (r^2) = \lim_{r\to0^+} \sin(\theta) \frac{\ln(r^2)}{1/r}$ = 0, since $|sin(\theta)| \leq 1$, where a change to polar coordinates was made, and L'Hospital rule was applied.

For the $\lim_{(x,y)\to(a,b)} f(x,y)$ to exist, the limit along any path containing (a, b) must be the same.

eg 47 $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \begin{cases} 0 & if\ \ y=0 \\ 1/2 & if\ \ y=x^2 \end{cases}$, so the limit does not exist.

eg 48 $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \begin{cases} 1 & if\ \ y=0 \\ -1 & if\ \ x=0 \end{cases}$, so the limit does not exist.

eg 49 $\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2+2y^2} = \begin{cases} 0 & if\ \ y=0 \\ 1/5 & if\ \ y=x \end{cases}$, so the limit does not exist.

Continuity at a Point

$f(x, y)$ is continuous at the point (x_0, y_0) if:

1. $f(x_0, y_0)$ is defined (no holes or jumps)

2. The limit at (x_0, y_0) exists

3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$

eg 50 $f(x, y) = xy^2 \sin(xy)$ is continuous at $\left(\frac{1}{2},\pi\right)$ since $\lim_{(x,y)\to\left(\frac{1}{2},\pi\right)} xy^2 \sin(xy) = \frac{\pi^2}{2}$.

eg 51 $f(x, y) = \frac{1}{x^2+2y^2}$ is discontinuous at (0, 0) since the function is not defined.

If $f(x, y, z)$ is continuous at the point (x_0, y_0, z_0), $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0)$.

eg 52 Since $\lim_{(x,y,z)\to(\frac{1}{2},\pi,2)} xy^2\cos(xyz) = -\frac{\pi^2}{2} = f(\frac{1}{2},\pi,2)$, $f(x,y,z) = xy^2\cos(xy/z)$ is continuous at the point $(\frac{1}{2},\pi,2)$.

eg 53 $f(x, y, z) = \frac{-1}{x^2+2y^2+z^2}$ is discontinuous at (0, 0, 0) since is not defined at that point.

Homework 12.6

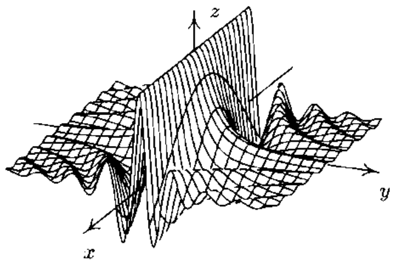
1. Use polar coordinates to find the following limits or show that they do not exist:

(a) $\lim_{(x,\,y)\to(0,\,0)} \frac{xy}{\sqrt{x^2+y^2}}$; (b) $\lim_{(x,\,y)\to(0,\,0)} \frac{x+y}{\sqrt{x^2+y^2}}$; (c) $\lim_{(x,\,y)\to(0,\,0)} \frac{(x^2+y^2)^2+x^3y^3}{(x^2+y^2)^2}$

2. Use polar coordinates to find the value of $\lim_{(x,\,y)\to(0,\,0)} \frac{y}{x}$ or show that the limit does not exist.

3. Show that $\lim_{(x,\,y)\to(0,\,0)} \frac{x^2y}{x^4+y^2}$ does not exist by considering (x, y) that approach (0, 0) along different parabolas.

4. The figure below shows the graph of $g(x, y) = \frac{10\cos(xy)}{1+2y^2}$. Find the global maximum of $g(x, y)$ and the values of (x, y) where it occurs.



5. Evaluate

a. $\lim_{(x,y)\to(0,0)} (x^2 + y^2)\ln(x^2 + y^2)$

b. $\lim_{(x,y)\to(0,0)} \frac{1-\sec^2(x^2+y^2)}{1-\cos(x^2+y^2)}$.

c. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

6. Evaluate by considering different paths of approach.

a. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}$

b. $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$

c. $\lim_{(x,y)\to(0,0)} \frac{x^2-3xy+y^2}{x^2+2y^2}$

d. $\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$

e. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$

f. $\lim_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2}$

7. Show that $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$ does not exist. Ans: $\begin{cases} 0 \; if \; x = 0 \\ 1 \; if \; x = y^2 \end{cases}$

8. Where is the function $f(x, y) = \frac{1}{x^2+y^2-1}$ discontinuous?

9. Where is the function $f(x, y) = \frac{1}{(x-3)^2+(y+2)^2}$ discontinuous?

Answers 12.5 1) 0, DNE, 1; 2) DNE; 4) *g(x,0)* =10; 5) 0,-2, 2; 7) $\begin{cases} 0 \; if \; x = 0 \\ 1 \; if \; x = y^2 \end{cases}$ 8) Along the unit circle; 9) at the point (3,-2).