Alternative Method for Expressing Rational Functions Into Partial Fractions

Case 1 Distinct Roots:

Consider

$$\frac{f(x)}{(x-a)(x-b)} \qquad \textbf{(1)}$$

where the degree of $f(x)$ is less than the degree of $(x-a)(x-b)$ where $\mathbf{a} \neq \mathbf{b}$. The partial fraction expansion will be given by

$$\frac{f(x)}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} \qquad (\mathbf{2})$$

where A and B must be determined. If we multiply $(\mathbf{2})$by $(x-a)$ we obtain

$\frac{f(x)}{(x-b)} = A + \frac{B(x-a)}{(x-b)}$. If we evaluate at $x = a$, we obtain $A = \frac{f(\mathbf{a})}{(\mathbf{a}-b)}$. This is the same as covering-up $(x-a)$ in **(1)** and evaluating at $x = a$.

If we multiply $(2)$by $(x-b)$ we obtain $\frac{f(x)}{(x-\mathbf{a})} = \frac{A(x-b)}{(x-a)} + B$. If we evaluate at $x = b$, we obtain $B = \frac{f(b)}{(b-a)}$. This is the same as covering-up $(x-b)$ in **(1)** and evaluating x at b.

This is what is known as the **Cover-up Method**

NOTE: If degree of the numerator is greater than or equal to the degree of the denominator, long division must be performed first and then partial fraction expansion can be applied to the remainder divided by the divisor part.

Example: Find the partial fraction expansion of $F(x) = \frac{x+1}{x(x-1)(x-2)}$

By partial fractions $\frac{x+1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-2)}$.

By the cover-up method $A = \frac{x+1}{(x-1)(x-2)}\Big|_0 = \frac{1}{2}$; $B = \frac{x+1}{x(x-2)}\Big|_1 = -2$;
$C = \frac{x+1}{x(x-1)}\Big|_2 = \frac{3}{2}$.

So the expansion of $\frac{x+1}{x(x-1)(x-2)} = \frac{1}{2x} - \frac{2}{(x-1)} + \frac{3}{2(x-2)}$

Case II Repeated Roots:

Consider $\frac{f(x)}{(x-a)^4}$ .where the degree of $f(x)$ is less than 4. The partial fraction expansion will be given by

$$\frac{f(x)}{(x-a)^4} = \frac{A}{(x-a)^4} + \frac{B}{(x-a)^3} + \frac{C}{(x-a)^2} + \frac{D}{(x-a)^1} \qquad (\mathbf{3})$$

with the repeated power factor written as terms in descending order and where A , B, C **and** D must be determined. If we multiply ($\mathbf{3}$)by $(x-a)^4$ we obtain

$$f(x) = A + B(x-a) + C(x-a)^2 + D(x-a)^3 \qquad (\mathbf{4})$$

If we evaluate at $x = a$, we obtain $f(a) = A$ (Cover-up method).

If we take the $first$ derivative of ($\mathbf{4}$)we obtain

$$f'(x) = 0 + B + 2C(x-a) + 3D(x-a)^2$$

If we evaluate at $x = a$, we obtain $f'(a) = B$

If we take the $second$ derivative of ($\mathbf{4}$)we obtain $f'(x) = 2C + 3 \times \mathbf{2}D(x-a)$

If we evaluate at $x = a$, we obtain $f''(a) = 2C$ **or** $C = f''(a)/2!$

If we take the $third$ derivative of ($\mathbf{4}$)we obtain $f'''(x) = 3 \times \mathbf{2}D$

If we evaluate at $x = a$, we obtain $f'''(a) = 3 \times \mathbf{2}D = 3!\, D$ **or** $D = f'''(a)/3!$

In the case where you have distinct linear and repeated factors, apply the Cover-up Method to the terms with distinct linear factors and the term with the highest power of the repeated factor. Find the other coefficients as described above by taking the derivative and dividing by the factorial .

Example: Find the Partial fraction expansion of $F(x) = \frac{x+1}{x(x-1)^3}$.

By partial fractions $\frac{x+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)}$

So $A = \frac{x+1}{(x-1)^3}|_0 = -1$; $B = \frac{x+1}{x}|_1 = 2$; $C = \frac{\left(\frac{x+1}{x}\right)'}{1!}|_1 = -1$; $D = \frac{\left(\frac{x+1}{x}\right)''}{2!}|_1 = 1$

or $\frac{x+1}{x(x-1)^3} = -\frac{1}{x} + \frac{2}{(x-1)^3} - \frac{1}{(x-1)^2} + \frac{1}{(x-1)}$

Case III Complex Roots (Irreducible):

Consider $\frac{f(x)}{g(x)(x^2+bx+c)}$ where the degree of $f(x)$ is less than the degree of $g(x)\left(x^2+bx+c\right)$and the roots of x^2+bx+c are complex of the form $a \pm bi.$ To solve this case treat the term $g(x)$ depending on its case and expand the irreducible part into $\frac{f(x)}{g(x)(x^2+bx+c)} = \frac{A}{g(x)} + \frac{Bx+C}{(x^2+bx+c)}$where B and C can be determined by using the cover-up method and evaluating both sides of the equation at either of the complex roots of x^2+bx+c**.**

Example: Find the Partial fraction expansion of $F(x) = \frac{x+1}{x(x^2+2x+2)}$.

$\frac{x+1}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{(x^2+2x+2)}$.

By cover-up on the first term we obtain

$A = \frac{x+1}{(x^2+2x+2)}|_0 = 1/2.$

By cover-up on the second term and evaluated at either root $-1 \pm i$ we obtain

$\frac{x+1}{x} \mid_{-1-i} = Bx + C|_{-1-i} \Rightarrow \frac{-i}{-1-i} = B(-1-i) + C \Rightarrow$

$\frac{-i\,(-1+i)}{2} = (-B+C) - Bi$ **or** $\frac{1}{2}+\frac{i}{2} = (-B+C) - Bi\,.$

If we equate real and imaginary parts, we obtain a system of two equations

$C - B = \frac{1}{2}$ and $-B = \frac{1}{2}$.

The solution of the system is $B = -\frac{1}{2}$ and $C = 0.$

The partial fraction expansion of $\frac{x+1}{x(x^2+2x+2)} = \frac{1}{2x} - \frac{x}{2(x^2+2x+2)}$.

Case IV Repeated Complex Roots

Consider $\frac{f(x)}{g(x)(x^2+bx+c)^2}$ where the degree of $f(x)$ is less than the degree of

$g(x)\left(x^2+bx+c\right)$and the roots of x^2+bx+c are complex of the form $a \pm bi$.
To solve this case treat the term $g(x)$ depending on its case and expand the complex

repeated part into $\frac{f(x)}{g(x)(x^2+bx+c)^2} = \frac{A}{g(x)} + \frac{Bx+C}{(x^2+bx+c)^2} + \frac{Dx+E}{(x^2+bx+c)}$ (**5**)
where $A,\ B$ and$C$ can be determined by using the cover-up method described above. To find **D** and **E,** we need to multiply (**5**) by $\left(x^2+bx+c\right)^2$**,** take the first derivative and evaluate at either root.

Example: Find the Partial fraction expansion of $F(x) = \frac{x+1}{x(x^2+1)^2}$.

$\frac{x+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)^2} + \frac{Dx+E}{(x^2+1)}$. We can find A,B,C by cover-up

$A = \frac{x+1}{(x^2+1)^2}|_0 = 1$. $Bx+C = \frac{x+1}{x}|_i$, so $B = -1, C = 1$.
D and E can be found by $\frac{d}{dx}[\frac{x+1}{x} = \frac{A\left(x^2+1\right)^2}{x} + Bx + C + (Dx+E)\left(x^2+1\right)\ ]|_i$, so $D = -1, E = 0$.

The partial fraction expansion of $\frac{x+1}{x(x^2+1)^2} = \frac{1}{x} - \frac{x-1}{(x^2+1)^2} - \frac{x}{(x^2+1)}$.

General Notes:
Depending on the function, methods including derivatives of rational functions (cases II and IV) could become algebraically involved . Some times is easier to use a combination of systems of equations and some of the above methods.