Infinite Sequences and Series

12.1 Infinite Sequences

An infinite sequence is a function whose domain is the set of non negative integers n.
So the infinite sequence $\{a_n\}_{n=n_0}^{\infty} = a_{n_0}, a_{n_0+1}, a_{n_0+2}, a_{n_0+3}, a_{n_0+4}, \ldots$ with n_0 also a non negative integer. The a_n's are called the terms of the sequence.

eg 1 The sequence $\{\frac{1}{n+1}\}_{n=0}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \cdots$

eg 2 The sequence $\{2n\}_{n=1}^{\infty}$ is the sequence of all positive even integers

eg 3 The sequence $\{2n+1\}_{n=0}^{\infty}$ is the sequence of all positive odd integers

eg 4 The sequence $\{cos\,(n\pi)\}_{n=0}^{\infty} = \{-1^n\}_{n=0}^{\infty}$

The limit of a sequence

The infinite sequence $\{a_n\}_{n=n_0}^{\infty}$ has the limit L i.e. $\left(\lim_{n\to\infty} a_n = L\right)$ if a_n gets arbitrarily close to L as n increases without bound.
This is $\{a_n\}_{n=n_0}^{\infty}$ has the limit L if given any ϵ>0 $\exists\, N$ such that $|a_n - L| < \epsilon$ for some $n \geq$ N.

If $\{a_n\}_{n=n_0}^{\infty}$ has the limit L, we say the sequence converges, otherwise the sequence diverges.
All the properties of limits of functions apply to limits of sequences.

eg 5 $\{\frac{n^2+1}{n+1}\}_{n=0}^{\infty} \to \lim_{n\to\infty} \frac{n^2+1}{n+1} = \infty$; $\{(-1)^n \frac{n-1}{n^2+1}\}_{n=0}^{\infty} \to \lim_{n\to\infty} (-1)^n \left(\frac{n-1}{n^2+1}\right) = 0.$
$\{n^2 e^{-n}\}_{n=0}^{\infty} \to \lim_{n\to\infty} \frac{n^2}{e^n} = 0.$
Note: Eliminating a finite numbers of terms of a sequence does not affect its convergence.

eg 6 Find the following limits of sequences
$\lim_{n\to\infty} \frac{ln(n)}{n} = 0$; $\lim_{n\to\infty} \sqrt[n]{n} = 1$; $\lim_{n\to\infty} r^{\frac{1}{n}} = 1$ for $r > 0$; $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$;
$\lim_{n\to\infty} r^n = 0$ for $|r| < 1$; $\lim_{n\to\infty} 1 + (-1)^n$ Diverges; $\lim_{n\to\infty} \sqrt{n-100} - \sqrt{n+100} = 0.$

Monotone Sequences

Strictly Monotone Sequence
If $a_{n+1} > a_n$ or $\frac{a_{n+1}}{a_n} > 1$, the sequence $\{a_n\}_{n=0}^{\infty}$ increases. eg. $\{\frac{n}{n+1}\}_{n=0}^{\infty}$
If $a_{n+1} < a_n$ or $\frac{a_{n+1}}{a_n} < 1$, the sequence $\{a_n\}_{n=0}^{\infty}$ decreases. eg. $\{\frac{1}{n}\}_{n=0}^{\infty}$
Monotone Sequence
If $a_{n+1} \geq a_n$ or $\frac{a_{n+1}}{a_n} \geq 1$, the sequence $\{a_n\}_{n=0}^{\infty}$ non decreases.
If $a_{n+1} \leq a_n$ or $\frac{a_{n+1}}{a_n} \leq 1$, the sequence $\{a_n\}_{n=0}^{\infty}$ non increases.

eg 7 $\{1+\frac{1}{n}\}_{n=1}^{\infty}$ is decreasing since $\left(1+\frac{1}{n+1}\right) - \left(1+\frac{1}{n}\right) = \frac{-1}{n(n+1)} < 0$ for $n \geq 1$.

eg 8 $\{\frac{n^n}{n\,!}\}_{n=1}^{\infty}$ is increasing since $\frac{n+1^{n+1}}{(n+1)\,!} / \frac{n^n}{n\,!} = \left(\frac{n+1}{n}\right)^n > 1$ for $n \geq 1$.

A sequence that increases and decreases is called non monotone.

eg 9 $\{\frac{n^2}{n\,!}\}_{n=1}^{\infty}$; $\frac{a_{n+1}}{a_n} = \frac{n+1}{n^2} > 1$ increases for $n = 1$; and $\frac{n+1}{n^2} < 1$ decreases for $n \geq 2$.
$\{2, \frac{3}{4}, \frac{1}{4} \cdots\}$

Another way to determine if a sequence is increasing or decreasing is if we can represent the sequence with a function, we can apply the first derivative test to the function.

eg 10 The sequence $\{\frac{ln(n+2)}{n+2}\}_{n=1}^{\infty}$is decreasing since $f'(x) = \frac{d}{dx}\frac{ln(x+2)}{x+2} = \frac{1-ln(x+2)}{(x+2)^2} < 0$ for $n \geq 1$.

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12.2 Series

A series is the sum of the terms of a sequence.

Finite Series

A finite series is the sum of the terms of a finite sequence.

The sequence of partial sum of the finite sequence $\{a_k\}_{k=1}^{n}$ is defined as $\{S_k\}_{k=1}^{n} = \mathrm{S}_1, \ldots, \mathrm{S}_n$ where $S_1 = a_1,\ S_2 = a_1 + a_2,\ \cdots,\ S_n = a_1 + a_2 + a_{3+\ldots+}a_n$.

S_n is called the n_{th} partial sum of the sequence where n is the number of terms in the series.

Consider the sequence $\{a_k\}_{k=1}^{n}$. Its corresponding series will be

$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n = S_n$, the n_{th} partial sum of the sequence

Example of a Finite Series; The Geometric Series

Consider the n_{th} partial sum of the Geometric Sequence $\{ar^{k-1}\}_{k=1}^{n}$

$S_n = \sum_{k=1}^{n} ar^{\mathrm{k}-1} = a{+}ar{+}ar^2{+}\cdots ar^{n-1}$. $S_n - rS_n = a - ar^n$ so

$S_n = \sum_{k=1}^{n} ar^{\mathrm{k}-1} = a\frac{1-r^n}{1-r}$ for $r \neq 1$

eg 11 Find $\sum_{k=1}^{5} 7\left(-\frac{2}{3}\right)^k$. Since $r = -\frac{2}{3}$ and $a = -\frac{14}{3}$, $S_5 = -\frac{14}{3}\left(\frac{1-\left(-\frac{2}{3}\right)^5}{1-\left(-\frac{2}{3}\right)}\right) = \frac{-770}{243}$.

eg 12 Example of finite series are:

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Infinite Series

Since $S_n = \sum_{k=1}^{n} a_k$ is n_{th} partial sum of the sequence $\{a_k\}_{k=1}^{\infty}$, the infinite series

$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_n + \cdots = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$.

If the this limit exists, then the limit is the sum of the infinite series, and we say

$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k = S$

Properties of Convergent Series

1) $\sum_{k=1}^{\infty} (a_k \pm b_k) = \sum_{k=1}^{\infty} a_k \pm \sum_{k=1}^{\infty} b_k$

2) $\sum_{k=1}^{\infty} c a_k = c \sum_{k=1}^{\infty} a_k$

3) Deleting a finite numbers of terms in a series will not affect its convergence.

Note:

The sum of most convergent infinite series can not be found. Here are some examples of series whose sums can be found.

Telescoping Series

eg 13 $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+1} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1.$

eg 14 Find $\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)} = \frac{1}{3}$

Infinite Geometric Series

Since $S_n = a\frac{1-r^n}{1-r}$ is the n_{th} partial sum of the Geometric Sequence will be $\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\frac{1-r^n}{1-r}$. This limit will only converge for $|r| < 1$ to $\frac{a}{1-r} = S$ the sum of the series.

eg 16 Suppose you want to go from point A to point B two miles away. If you first go half the distance, then go half that distance, and so forth, so that at each stage you go half as far as you did in the previous stage, your total distance will be $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 2$

If we write the sum as $1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\cdots = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{(1-\frac{1}{2})} = 2$ since this is a Geometric Series with $a = 1$ and $r = \frac{1}{2} < 1$.

eg 17 $\sum_{k=1}^{\infty} \frac{7}{(-6)^{k-1}} = 7\frac{1}{1+\frac{1}{6}} = 6$ since $a = 7$ and $r = -1/6$

eg 18 $\sum_{k=1}^{\infty} \frac{2}{(-3)^k} = \frac{-2}{3}\frac{1}{1+\frac{1}{3}} = -1/2$ since $a = \frac{-2}{3}$(the first term in the series) and $r = -1/3$

eg 19 $\sum_{k=0}^{\infty} \frac{2^{k+2}}{3^{k-1}} = \frac{12}{(1-\frac{2}{3})} = 36$ since $a = 12$ and $r = \frac{2}{3} < 1$.

eg 20 $\sum_{k=1}^{\infty} \left(-\frac{3}{2}\right)^k =$ diverges since $|-\frac{3}{2}| > 1$

eg 21 Express $0.4444\overline{4}$ as a fraction by using a geometric series

Since $0.4444\overline{4} = 0.4 + 0.04 + 0.004 + \cdots = \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} \cdots = \sum_{k=1}^{\infty} 4\left(\frac{1}{10}\right)^k =$ $\frac{4}{10}\frac{1}{1-\frac{1}{10}} = \frac{4}{9}$.

eg 22 Find the values of x the will make $\sum_{k=0}^{\infty} x^k$ a convergent series. $\frac{1}{1-x}$; $|x| < 1$.

eg 23 Find the values of x the will make $\sum_{k=1}^{\infty} \frac{3^k}{x^{k+1}}$ a convergent series.

$\sum_{k=0}^{\infty} \frac{3^k}{x^{k+1}} = \frac{1}{x}\frac{1}{1-\frac{3}{x}} = \frac{1}{x-3}$ for $|\frac{3}{x}| < 1$; $|\frac{x}{3}| > 1$; $|x| > 3$.

eg 24 A ball dropped from a height of $4m$. rebound each time 75% of its previous height. Find the total distance traveled by the ball.

The total distance will be $D = 4 + 2[4(.75) + 4(.75)^2 \cdots] = 4 + 2\sum_{k=1}^{\infty} 4(\frac{3}{4})^k = 4 + 2\frac{3}{1-\frac{3}{4}} = 28m$

The Divergence Test (DT)

If $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$

Proof: If $\sum a_k$ converges then it will have a limit L .Since $S_n - S_{n-1} = a_k$,

$$\lim_{k\to\infty} a_k = \lim_{n\to\infty} S_n - S_{n-1} = \lim_{n\to\infty} \left(\sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k \right) = L - L = 0.$$

The contrapositive of this statement is known as the divergence test.

If $\lim_{k\to\infty} a_k \neq 0$, then $\sum a_k$ diverges.

eg 25 $\sum_{k=1}^{\infty} \frac{2k^2-1}{1-k}$; $\lim_{k\to\infty} \frac{2k^2-1}{1-k} = -2 \neq 0$ (diverges)

eg 26 $\sum_{k=1}^{\infty} \left(\frac{k+1}{k-1}\right)^k$; $\lim_{k\to\infty} \left(\frac{k+1}{k-1}\right)^k = e^2 \neq 0$ (diverges)

eg 27 $\sum_{k=1}^{\infty} ln\left(\frac{1}{k}\right)$; $\lim_{k\to\infty} ln\left(\frac{1}{k}\right) = \infty \neq 0$ (diverges)

eg 28 $\sum_{k=1}^{\infty} cos(k\pi)$; $\lim_{k\to\infty} cos(k\pi) =$ DNE (diverges)

eg 29 $\sum_{k=1}^{\infty} 2^{(1/nk)}$; $\lim_{k\to\infty} 2^{(1/k)} = 1 \neq 0$ (diverges)

eg 30 $\sum_{k=1}^{\infty} cos(1/k)$; $\lim_{k\to\infty} cos(1/k) = 1 \neq 0$ (diverges)

12.3 The Integral Test (IT)

Let $\sum_{k=1}^{\infty} a_k$ be a series with non negative terms, and let $f(x)$ be the function that results when k is replace by x in the formula for a_k. If $f(x)$ is non increasing and continuous for $x \geq 1$, then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ both converge or diverge.

NOTES:

Use when $f(x)$ is easy to integrate.

If the converges, series do not converges to the value of the integral

eg 31 $\sum_{k=1}^{\infty} ke^{-2k} \Rightarrow \int_1^{\infty} xe^{-2x}dx = \lim_{b\to\infty} -\frac{e^{-2b}}{2}\left(b-\frac{1}{2}\right)+\frac{3e^{-2}}{4} = \frac{3e^{-2}}{4}$. Since the integral converges, the series converges.

eg 32 $\sum_{k=1}^{\infty} \frac{2k}{k^2+1} \Rightarrow \int_1^{\infty} \frac{2x}{x^2+1}dx = \lim_{b\to\infty} ln\left(b^2+1\right) - ln(1+1) = \infty$. Since the integral diverges, the series diverges.

eg 33 $\sum_{k=1}^{\infty} \frac{1}{k} \Rightarrow \int_1^{\infty} \frac{1}{x}dx = \lim_{b\to\infty} ln(b) - ln(1) = \infty$. Since the integral diverges, the series diverges.

p-series

$\sum_{k=1}^{\infty} \frac{1}{k^p} \Rightarrow \int_1^{\infty} \frac{1}{x^p}dx = \lim_{b\to\infty} \frac{x^{-p+1}}{1-p}\Big|_1^b = \lim_{b\to\infty} \frac{b^{-p+1}}{1-p} - \frac{1}{1-p} = \frac{1}{1-p}$ when $p > 1$. Since the integral converges for $p > 1$, the series converges for $p > 1$.

eg 34 $\sum_{k=1}^{\infty} \frac{1}{(k+5)^3}$ let $j = k+5 \Rightarrow \sum_{j=6}^{\infty} \frac{1}{j^3}$ converges $p > 1$

NOTE: If $\lim_{k\to\infty} a_k = 0$, then $\sum a_k$ may or may not converge.

eg 35 $\sum_{k=1}^{\infty} \frac{1}{k}$; $\lim_{k\to\infty} \frac{1}{k} = 0$, and the Harmonic series diverges by the Integral Test.

eg 36 $\sum_{k=1}^{\infty} \frac{1}{k^3}$; $\lim_{k\to\infty} \frac{1}{k^3} = 0$, and the p-series $(p > 1)$converges by the Integral Test.

Estimating the Sum of a Series (Integral Test)

Assume $\sum_{k=1}^{\infty} a_k$ be a convergent series by the integral tests, and let $f(x)$ be the function that results when k is replace by x. The reminder R_n is the error made when S_n (the n_{th} partial sum) is used as an approximation of the total sum. If $R_n = \sum_{k=n+1}^{\infty} a_k$ is defined to be $S - S_n$ (the sum after the n_{th} term), then $\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$

eg 37 Find the error involved to approximate $\sum_{k=1}^{\infty} \frac{1}{k^2}$ by adding the first five terms.

$S_5 \approx 1.46361$; the error $R_5 \le \int_5^{\infty} \frac{1}{x^2} dx = \lim_{a\to\infty} \int_5^{a} \frac{1}{x^2} dx = \frac{1}{5} = 0.2$

So $\sum_{k=1}^{\infty} \frac{1}{k^2} \approx 1$

eg 38 How many terms of $\sum_{k=1}^{\infty} \frac{1}{x^2}$ are needed to obtain two-decimal place accuracy (an error $< .005$)?

Since $\int_n^{\infty} \frac{1}{x^2} dx < .005$ or $\frac{1}{n} < .005$ so $n > 200$.

What this means is that $\sum_{k=1}^{201} \frac{1}{k^2} \approx 1.63995$ is an approximation of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ to two decimal places.

Since $R_n = S - S_n$, if we rearrange $\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$ we obtain

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le S_n + \int_{n}^{\infty} f(x)\,dx.$$

eg 39 Find the error involved to approximate $\sum_{k=1}^{\infty} \frac{1}{k^2}$ by adding the first five terms using the previous formula.

$S_5 + \int_6^{\infty} \frac{1}{x^2}\,dx \le S \le S_5 + \int_5^{\infty} f(x)\,dx$ or $1.46361 + \frac{1}{6} \le S \le 1.46361 + \frac{1}{5}$. Since S is the midpoint of the interval, the error is at most half the length of the interval.

So $S = \sum_{k=1}^{\infty} \frac{1}{k^2} \approx 1.64694$ with error $\le .005$. We were able to approximate $\sum_{k=1}^{\infty} \frac{1}{k^2}$ by only computing 5 terms instead of 200 terms of the last example.

12.4 The comparison Tests

Comparison Test (CT)
Let $\sum a_k$ and $\sum b_k$ be series with non negative terms such that $a_k \le b_k$.

If $\sum b_k$ converges then $\sum a_k$ converges, and if $\sum a_k$ diverges then $\sum b_k$ diverges.

NOTE: Use this test as a last resort. Some other tests are easier to use.

eg 40 $\sum_{k=1}^{\infty} \frac{1}{(k+1)^2}$; Since $\frac{1}{(k+1)^2} < \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges $(p > 1)$, then $\sum_{k=1}^{\infty} \frac{1}{(k+1)^2}$ converges.

eg 41 $\sum_{k=1}^{\infty} \frac{1}{2^k+10}$; Since $\frac{1}{2^k+10} < \frac{1}{2^k}$ and $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges $\left(\text{Geo. } r = \frac{1}{2}\right)$, then $\sum_{k=1}^{\infty} \frac{1}{2^k+10}$ converges.

eg 42 $\sum_{k=1}^{\infty} \frac{1}{2^k-10}$; Since $\frac{1}{2^k-10} < \frac{100}{2^k}$ for $k > 3$, and $\sum_{k=1}^{\infty} \frac{100}{2^k}$ converges $\left(\text{Geo. } r = \frac{1}{2}\right)$, then $\sum_{k=1}^{\infty} \frac{1}{2^k-10}$ converges.

eg 43 $\sum_{k=1}^{\infty} \frac{k^5-4k^2+x+1}{2k^6-4k^3+2^2}$; Since $\frac{2k^5-4k^2+k+1}{k^6-4k^3+k^2} > \frac{1}{k}$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges $(p < 1)$ then $\sum_{k=1}^{\infty} \frac{k^5-4k^2+k+1}{2k^6-4k^3+k^2}$ diverges.

eg 44 $\sum_{k=0}^{\infty} \frac{9+sin(k)}{2^k}$; Since $\frac{9+sin(k)}{2^k} \le \frac{10}{2^k}$ and $\sum_{k=1}^{\infty} \frac{10}{2^k}$ converges $\left(\text{Geo. } r = \frac{1}{2}\right)$, then $\sum_{k=0}^{\infty} \frac{9+sin(k)}{2^k}$ converges.

Estimating Sums:

eg 45 Find the error involved to approximate $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ by adding the first fifty terms.

$S_{50} = \sum_{k=1}^{\infty} \frac{1}{k^2+1} \approx 3.51881$. Since $\sum_{k=1}^{\infty} \frac{1}{k^2+1} \le \sum_{k=1}^{\infty} \frac{1}{k^2}$, the remainder after 50 terms $R_{50} \le T_{50} = \int_{50}^{\infty} \frac{1}{x^2} dx = \frac{1}{50} = 0.02$, where T_{50} is the remainder of the test series.

eg 46 How many terms are needed to estimate $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ with an error $< .0005$?

Since $\sum_{k=1}^{\infty} \frac{1}{k^2+1} \le \sum_{k=1}^{\infty} \frac{1}{k^2}$, $R_n = \sum_{k=n+1}^{\infty} \frac{1}{k^2+1} \le T_n = \sum_{k=n+1}^{\infty} \frac{1}{k^2}$. To find estimate $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ with an error $< .0005$, it suffice to find $\sum_{k=1}^{\infty} \frac{1}{k^2}$ with an error $< .0005$. So $R_n \le \int_n^{\infty} \frac{1}{x^2} dx \le .0005$ or $\frac{1}{n} < .0005$ or $n > 2000$. This means $S_{2000} = 1.076$ with error $< .0005$; or $S_{2000} \approx 1.076$.

Limit Comparison Test (LCT)

Let $\sum a_k$ and $\sum b_k$ be series with non negative terms such that $\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$

If $0 < \rho < +\infty$, then both series converge or diverge. If $\rho = 0$ or $\rho = \infty$, the test fails.

NOTE: Require some skill in choosing the series $\sum b_k$ for comparison.

eg 47 $\sum_{k=1}^{\infty} \frac{1}{9k+6}$; Since $\frac{1}{9k+6}$ is not $> \frac{1}{k}$, a comparison test can not be made with $\sum_{k=1}^{\infty} \frac{1}{k}$. If we let $a_k = \frac{1}{9k+6}$ and $b_k = \frac{1}{k}$, $\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \rho = \lim_{k\to\infty} \frac{k}{9k+6} = \frac{1}{9}$, $(0 < \rho < +\infty)$.

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (harmonic; $p = 1$), $\sum_{k=1}^{\infty} \frac{1}{9k+6}$ diverges.

eg 48 $\sum_{k=1}^{\infty} \frac{10}{9^k+1}$; Since $\frac{10}{9^k+1}$ is not $< \frac{1}{9^k}$, a comparison test can not be made with $\sum_{k=1}^{\infty} \frac{1}{9^k}$. If we let $a_k = \frac{10}{9^k+1}$ and $b_k = \frac{1}{9^k}$, $\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \rho = \lim_{k\to\infty} \frac{10\times 9^k}{9^k+1} = 10$,

$(0 < \rho < +\infty)$. Since $\sum_{k=1}^{\infty} \frac{1}{9^k}$ converges $\left(\text{Geo. } r = \frac{1}{9}\right)$, $\sum_{k=1}^{\infty} \frac{1}{9k+6}$ converges.

eg 49 $\sum_{k=1}^{\infty} \left(\frac{20+k}{3k+9}\right)^k$; A LCT with $\sum_{k=1}^{\infty} \frac{1}{3^k}$ will give $\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \left(\frac{3}{3}\,\frac{20+k}{k+3}\right)^k =$ $exp\left(\lim_{k\to\infty} k ln\left(\frac{20+k}{k+3}\right)\right) = e^{17}$. Since $\sum_{k=1}^{\infty} \frac{1}{3^k}$converges Geo. $\sum_{k=1}^{\infty} \left(\frac{20+k}{3k+9}\right)^k$converges.

eg 50 $\sum_{k=1}^{\infty} \frac{3^k+2^k}{5^k}$; Since $\frac{3^k+2^k}{5^k}$ is not $< \frac{3^k}{5^k}$, a comparison test can not be made with $\sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^k$. If we let $a_k = \frac{3^k+2^k}{5^k}$ and $b_k = \frac{3^k}{5^k}$, $\rho = \lim_{k\to\infty} 1 + \left(\frac{2}{3}\right)^k = 1$.

Since $\sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^k$ converges $\left(\text{Geo. } r = \frac{3}{5}\right)$, $\sum_{k=1}^{\infty} \frac{3^k+2^k}{5^k}$ converges.

12.5 The Alternating Series Test (AST)

An alternating series is a series of the form $\sum_k (-1)^k a_k$ or $\sum_k (-1)^{k+1} a_k$.

If $a_k > 0$ for all k, the series $\sum_k (-1)^k a_k$ or $\sum_k (-1)^{k+1} a_k$ converges if

a) a_k is non increasing b) $\lim_{k\to\infty} a_k = 0$.

eg 51 $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ is called the alternating harmonic series.

Since $a_k > 0$ for all k and $\lim_{k\to\infty} \frac{1}{k} = 0$, the series converges.

Note that the harmonic series diverges.

eg 52 Consider $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+4}{k^2+k}$. Since $\frac{d}{dk}\frac{k+4}{k^2+k} = \frac{-2k^2-8k-3}{(k^2+k)^2} < 0$ for all k (decreasing) and $\lim_{k\to\infty}\frac{k+4}{k^2+k} = 0$, the series converges by AST.

eg 53 Consider $\sum_{k=3}^{\infty}(-1)^{k}\frac{ln(k)}{k}$. Since $\frac{d}{dk}\frac{ln(k)}{k} = \frac{1-ln(k)}{k^2} < 0$ for all $k > 3$ (decreasing) and since $\lim_{k\to\infty}\frac{ln(k)}{k} = 0$, the series converges by AST.

Note:
If condition (b) fails, $\lim_{k\to\infty} a_k \neq 0$, the series diverges by the divergence test.
If condition (a) fails, series of absolute values is not monotone, no conclusion can be made.

Estimation of Error in the Alternating Series Test

If $S = \sum_k (-1)^k a_k$ is the sum of an alt. series that satisfy the two condition of the AST, the error in using s_n to approximate the sum S is $|R_n| = |S - s_n| \leq |s_{n+1} - s_n| = a_{n+1}$.
This means that the error of the sum after the n_{th} term (the reminder after n) is smaller than the first neglected term.

eg 54 Consider $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}$. Since $S_{100} \approx 0.9015422$, and $|\mathrm{S} - s_{100}| \leq \frac{1}{101^3} \approx 0.00001$, S_{100} lies within 0.00001of the true limit S, or S lies between S_{100} and S_{101}

eg 55 Find n such that $|R_n| < .01$ for $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$.
Since $a_n = \frac{1}{k+1} < .01 \rightarrow k > 100$, $(a_{101} = \frac{1}{101} < .01)$, and since $|\mathrm{S} - s_{100}| \leq a_{101} < .01$, $\sum_{k=1}^{100}\frac{(-1)^{k+1}}{k} \approx 0.688172$ with error $< .01$.
Later we will show that $S = ln(2)$. So $|ln(2) - s_{100}| \approx 0.005 < .01$

eg 56 Find $\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}$ correct to three decimal places, this is ,with an error less than 0.0005.
If we try several values of k, $\frac{1}{5!} \approx .008$; $\frac{1}{6!} \approx .001$; $\frac{1}{7!} \approx .000198$. Since $|\mathrm{S} - s_6| \leq a_7 = .000198 < .0002$, $\sum_{k=0}^{6}\frac{(-1)^k}{k!} \approx .368$ correct to three decimal places.

12.6 Absolute convergence and the Ratio and Root Tests
Absolute Convergence (AS) and Conditional Convergence (CC)

Let $\sum u_k$ have some positive and some negative terms. If $\sum |u_k|$ converges then $\sum u_k$ converges absolutely (in absolute value). If $\sum u_k$ converges but $\sum |u_k|$ diverges then $\sum u_k$ converges conditionally (not in Absolute value).

eg 57 $\sum_{k=1}^{\infty} \frac{cos(k)}{k^2}$ converge absolutely since by CT, $|\frac{cos(k)}{k^2}| < \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges $(p = 2 > 1)$.

Consider the alternating series $\sum_k (-1)^k a_k$ with $a_k \geq 0$.

If the series $\sum_k (-1)^k a_k$:
- Conv. by AST
  - if $\sum_k a_k$ conv. → the series conv. absolutely
  - if $\sum_k a_k$ div. → the series conv. conditionally
- Div. by DT → Diverges

eg 58 $\sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{3}\right)^k$ converges absolutely since $\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k$ converges (Geo. $|r| = \frac{1}{3} < 1$);

eg 59 Since $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ converges by AST but $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges $(p = 1)$, the series $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ converges conditionally.

eg 60 $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ converges by AST since a_k is non increasing and $lim \to 0$. But since $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges $\left(p = \frac{1}{2} < 1\right)$, the series converges conditionally.

eg 61 $\sum_{k=1}^{\infty} \frac{cos(k\pi)}{ln(k+1)}$ converges by AST since a_k is non increasing and $lim \to 0$. But since $\sum_{k=1}^{\infty} \frac{(-1)^k}{ln(k+1)}$ diverges $\left(\text{by comparison with} \sum_{k=1}^{\infty} \frac{1}{k}\right)$, the series converges conditionally.

eg 62 $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{ln(k)}{3k+1}$ converges by AST since a_k is non increasing and $lim \to 0$. But since $\sum_{k=1}^{\infty} \frac{ln(k)}{3k+1}$ diverges $\left(\text{by comparison with} \sum_{k=1}^{\infty} \frac{1}{3k}\right)$, the series converges conditionally.

Alternating p-series

$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^p}$ converges conditionally for $0 < p \leq 1$ and absolutely for $p > 1$

<u>Rearranging terms of an alternating series</u>

The terms of an absolute convergent series may be rearranged without changing the sum of the series.

The terms of any conditional convergent series may be rearrange to diverge or to converge to any desired sum.

Consider the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots = S$.

Later on we will show that $S = ln(2)$
If we rearrange the terms of the series
$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{8}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} \cdots = .$
$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots\right) = \frac{1}{2}S.$
If we rearrange the terms of the series this way
$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} - \frac{1}{6} + \frac{1}{11} + \frac{1}{13} - \frac{1}{8} + \frac{1}{15} + \frac{1}{17} - \frac{1}{10} + \frac{1}{19} + \frac{1}{21} - \frac{1}{12} + \frac{1}{23} + \frac{1}{25} \cdots,$
and start adding terms, we see that after adding the positive terms, we get numbers that get closer to 1^+ as $k \to \infty$, and after adding a negative term, we get numbers that get closer to 1^- as $k \to \infty$. With this we can conclude that the sum $S = 1$.

Ratio Test for Absolute Convergence (RT)

Let $\sum u_k$ be any series such that $\rho = \lim_{k \to \infty} \frac{|u_{k+1}|}{|u_k|}$
a) the series converges absolutely if $\rho < 1$.
b) the series diverges if $\rho > 1$ or $\rho = \infty$.
c) No conclusion if $\rho = 1$. (The series can converge absolutely, conditionally or diverge)
Note: Try this when u_k involve factorials and k_{th} powers

eg 63 Consider $\sum_{k=1}^{\infty} (-1)^k \frac{k}{5^k}$. Since $\rho = \lim_{k \to \infty} \frac{|u_{k+1}|}{|u_k|} = \frac{1}{5} < 1$, the series converges absolutely.
eg 64 Consider $\sum_{k=1}^{\infty} k! e^{-k}$. Since $\rho = \lim_{k \to \infty} \frac{|u_{k+1}|}{|u_k|} = \infty$, the series diverges.
eg 65 Consider $\sum_{k=1}^{\infty} \frac{(k+1)(k+2)}{k!}$. Since $\rho = \lim_{k \to \infty} \frac{|u_{k+1}|}{|u_k|} = 0 < 1$, the series converges absolutely.
eg 66 Consider $\sum_{k=1}^{\infty} (-1)^k \frac{k^2}{k^3+1}$. Since $\rho = \lim_{k \to \infty} \frac{|u_{k+1}|}{|u_k|} = 1$, the test fails.
Since $\frac{k^2}{k^3+1}$ is decreasing and $\lim_{k \to \infty} \frac{k^2}{k^3+1} = 0$, the series converges by AST. Since $\frac{k^2}{k^3+1} > \frac{1}{10k}$ and $\sum_{k=1}^{\infty} \frac{1}{10k}$ diverges (harmonic $p = 1$), then $\sum_{k=1}^{\infty} (-1)^k \frac{k^2}{k^3+1}$ converges conditionally.
eg. Below are two series with different convergence where the ratio test fails.
$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ converges cond. with $\rho = 1$; $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$ converges abs. with $\rho = 1$

Root Test (Root)
Let $\sum u_k$ be any series such that $\rho = \lim_{k \to \infty} \sqrt[k]{|u_k|}$
a) the series converges absolutely if $\rho < 1$.
b) the series diverges if $\rho > 1$ or $\rho = \infty$.
c) No conclusion if $\rho = 1$.
Note: Try this when u_k involve k_{th} roots.

eg 67 Consider $\sum_{k=1}^{\infty} k^k$; Since $\rho = \lim_{k\to\infty} \sqrt[k]{k^k} = \infty$, the series $\sum_{k=1}^{\infty} k^k$ diverges by the Root Test.

eg 68 Consider $\sum_{k=1}^{\infty} k^{-k}$; Since $\rho = \lim_{k\to\infty} \sqrt[k]{k^{-k}} = 0$, the series $\sum_{k=1}^{\infty} k^{-k}$ converges by the Root Test.

eg 69 Consider $\sum_{k=1}^{\infty} \frac{(ln(k))^k}{k^k}$. Since $\rho = \lim_{k\to\infty} \sqrt[k]{\frac{(ln(k))^k}{k^k}} = \lim_{k\to\infty} \frac{ln(k)}{k} = 0 < 1$, the series $\sum_{k=1}^{\infty} \frac{(ln(k))^k}{k^k}$ converges by the Root Test.

eg 70 Consider $\sum_{k=1}^{\infty} \frac{k!^k}{(k^k)^2}$ Since $\rho = \lim_{k\to\infty} \sqrt[k]{\frac{k!^k}{(k^k)^2}} = \lim_{k\to\infty} \frac{k!}{k^2} = \infty$, the series $\sum_{k=1}^{\infty} \frac{k!^k}{(k^k)^2}$ diverges by the Root Test

eg 71 Consider $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$ Since $\rho = \lim_{k\to\infty} \sqrt[k]{\frac{k^2}{2^k}} = \lim_{k\to\infty} \frac{\left(\sqrt[k]{k}\right)^2}{2} = \frac{1}{2} < 1$, the series $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$ converges by the Root Test.

12.8 Power Series

A power series is an infinite series whose terms are variables (powers of x)

$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

Any power series : a) converges for $x = 0$ b) converges absolutely for all x.
c) converges absolutely for some radius R (radius of convergence)such that $|x| < R$ were the convergence at the end points of $(-R, R)$ (interval of convergence) must be determined.

eg 72 For $\sum_{k=0}^{\infty} \frac{k!}{2^k} x^k$,$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} |\frac{k+1}{2} x| = \infty$, so the series converges for $x = 0$.

eg 73 $\sum_{k=0}^{\infty} \frac{x^k}{k!}$, $\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} |\frac{1}{k+1} x| = 0$, so the series converges abs. for all x.

eg 74 For $\sum_{k=0}^{\infty} \frac{(-2)^{k+1}}{k+1} x^{k+1}$, $\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} |-2\frac{k+1}{k+2} x| = 2|x|$ must converge for $|x| < \frac{1}{2}$. When $x = \frac{1}{2}$we have $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+1} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ that converges conditionally (alternating harmonic) ; when $x = -\frac{1}{2}$we have $\sum_{k=0}^{\infty} \frac{1}{k+1} = \sum_{k=1}^{\infty} \frac{1}{k}$ that diverges(harmonic). The radius of convergence is $R = \frac{1}{2}$ and the interval $(-\frac{1}{2}, \frac{1}{2}]$.

eg 75 For $\sum_{k=0}^{\infty} \frac{x^k}{k^2+1}$, $\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = |x|$ must converge for $|x| < 1$. When $x = 1$we have $\sum_{k=0}^{\infty} \frac{1}{k^2+1}$ converges (comparison with $\sum_{k=1}^{\infty} \frac{1}{k^2}$) ; when $x = -1$ we have $\sum_{k=0}^{\infty} \frac{(-1)^k}{k^2+1}$ converges because the one in absolute value converges (absolutely) with radius of convergence is $R = 1$ and interval of convergence $-1 \leq x \leq 1$.

eg 76 For $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}} x^k$, $\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = |x|$ must converge for $|x| < 1$. When $x = 1$we have $\sum_{k=1}^{\infty} \frac{-(-1)^k}{\sqrt{k}}$ that converges by AST but diverges in absolute value ($p = \frac{1}{2} < 1$), so it converges conditionally ; when $x = -1$we have $\sum_{k=1}^{\infty} \frac{-1}{\sqrt{k}}$ that diverges ($p = \frac{1}{2} < 1$) with radius of convergence is $R = 1$ and interval of convergence $(-1, 1]$.

eg 77 For $\sum_{k=1}^{\infty} \frac{5^k}{k^2} x^k$, $\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = 5\,|x|$ must converge for $|x| < \frac{1}{5}$. When $x = \frac{1}{5}$we have $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges ($p = 2 > 1$) ; when $x = -1$ we have $\frac{1}{5}\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges because the one in absolute value converges (absolutely) with radius of convergence is $R = \frac{1}{5}$ and interval of convergence is $(-\frac{1}{5}, \frac{1}{5}]$.

Power Series in $(x - a)$

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Theorem
Any power series :
a) converges for $x = a$
b) converges absolutely for all x.
c) converges absolutely for some radius R (radius of convergence centered at a)such that $|x - a| < R$ were the convergence at the end points of $(a - R, a + R)$
(interval of convergence) must be determined.

eg 78 For $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2} (x-4)^k$, $\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = |x - 4|$ must converge for $|x - 4| < 1$ or $3 < x < 5$. When $x = 3$ we have $\sum_{k=0}^{\infty} \frac{1}{(k+1)^2}$ converges (CT with$\sum_{k=1}^{\infty} \frac{1}{k^2}$) ; when $x = 5$ we have $\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2}$ converges because the one in absolute value converges (absolutely) with radius of convergence is $R = 1$ and interval of convergence is$[3, 5]$.

eg 79 For $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x+1)^k$, $\rho = \lim\limits_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = |x+1|$ must converge for $|x+1| < 1$ or $-2 < x < 0$. When $x = -2$ we have $-\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges (harmonic) ; when $x = 0$ we have $-\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges conditionally (alternating harmonic) with radius of convergence is $R = 1$ and interval of convergence is $(-2, 0]$.

eg 80 For $\sum\limits_{k=0}^{\infty} \frac{e^k}{(2k+1)!}(x-1)^k$, $\rho = \lim\limits_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim\limits_{k\to\infty} \left| \frac{e(k+1)}{(2k+3)(2k+2)} \right| = 0 < 1$ converges for all x with radius of convergence is $R = \infty$ and interval of convergence is $(-\infty, \infty)$.

12.9 Representation of Functions as Power Series

eg. $\sum\limits_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$ is a geometric series with $r = x$.

If we use the RT we find that $\rho = \lim\limits_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim\limits_{k\to\infty} \frac{|x^{k+1}|}{|x^k|} = |x|$ must converge for $|x| < 1$. When $x = 1$we have $\sum\limits_{k=0}^{\infty} 1$ that diverges to ∞; when $x = -1$we have $\sum\limits_{k=0}^{\infty} (-1)^k$ that diverges without a limit. The radius of convergence is $R = 1$ and the interval $-1 < x < 1$. This is the same result we expect from the geometric series with $r = x$.

We can say $\sum\limits_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ for $|x| < 1$.

eg 81 $\frac{x}{1-x} = x \sum\limits_{k=0}^{\infty} x^k = \sum\limits_{k=0}^{\infty} x^{k+1}$ for $|x| < 1$.

eg 82 $\frac{1}{1-x^2} = \sum\limits_{k=0}^{\infty} x^{2k}$ for $|x^2| < 1$ or $|x| < 1$.

eg 83 $\frac{1}{1+x} = \sum\limits_{k=0}^{\infty} (-x)^k = \sum\limits_{k=0}^{\infty} (-1)^k x^k$ for $|x| < 1$.

eg 84 $\frac{1}{2+x} = \frac{1}{2}\left(\frac{1}{1+\frac{x}{2}}\right) = \frac{1}{2}\sum\limits_{k=0}^{\infty} \left(-\frac{x}{2}\right)^k = \sum\limits_{k=0}^{\infty} (-1)^k \frac{x^k}{2^{k+1}}$ for $\left|\frac{-x}{2}\right| < 1$ or $|x| < 2$.

Theorem:

If $\sum\limits_{k=0}^{\infty} c_k (x-a)^k$ converges in $(a-R, a+R)$ for some $R > 0$, it will define a function $f(x) = \sum\limits_{k=0}^{\infty} c_k (x-a)^k$ in $(a-R, a+R)$. Such function *f* has derivatives of all orders inside the interval of convergence, and we can integrate and differentiate the series term wise with the same interval of convergence as the one of the original series.

eg 85 $\frac{1}{(1+x)^2} = \frac{d}{dx}\frac{-1}{(1+x)} = \frac{d}{dx}\sum_{k=0}^{\infty}(-1)^{k+1}x^k = \sum_{k=1}^{\infty}(-1)^{k+1}k\,x^{k-1} =$
$\sum_{k=0}^{\infty}(-1)^k\,(k+1)x^k$ for $|x| < 1.$

eg 86 $tan^{-1}(x) = \int \frac{1}{1+x^2}dx = \int \sum_{k=0}^{\infty}\left(-x^2\right)^k dx = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1} + c.$

If we let $x = 0, c = 0$. So $tan^{-1}(x) = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$ for $|x| \le 1,$ where we can show that the series converges at the end points.

eg 87 $ln(1-x) = \int \frac{-1}{1-x}dx = \int -\sum_{k=0}^{\infty}(x)^k dx = -\sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1} + c \overset{x=0}{\rightarrow}$
$-\sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1} = -\sum_{k=1}^{\infty}\frac{x^k}{k}$ for $-1 \le x < 1.$

If we let $x = \frac{1}{2},\ ln\left(1-\frac{1}{2}\right) = -ln(2) = -\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2}\right)^k}{k}$ or $ln(2) = \sum_{k=1}^{\infty}\frac{1}{k2^k}.$

Note: Term-by-term differentiation might not work for other type of series.

$\sum_{k=1}^{\infty}\frac{sin(k!x)}{k^2}$ converges while $\frac{d}{dx}\sum_{k=1}^{\infty}\frac{sin(k!x)}{k^2} = \sum_{k=1}^{\infty}\frac{k!cos(k!x)}{k^2}$ diverges.

12.10 Taylor and Maclaurin Series

A function that has all orders derivatives in an interval I can be expressed as a power series on I.
Lets assume that $f(x)$ is the sum of the power series centered at a

$$f(x) = \sum_{k=0}^{\infty}c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots.$$

Since $f(a) = c_0; f'(a) = c_1; f''(a) = 2c_2;\ f'''(a) = 2.3c_3; \cdots; f^n(a) = n!c_n,$
the series becomes

$$f(x) = \sum_{k=0}^{\infty}\frac{f^k(x)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots.$$

Definition:
Let f be a function with derivatives of all orders in an interval I containing a.
The Taylor Series generated by f at $x = a$ is

$$f(x) = \sum_{k=0}^{\infty}\frac{f^k(x)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots$$

The Taylor Series generated by f at $x = 0$ (Maclaurin Series) is

$$f(x) = \sum_{k=0}^{\infty}\frac{f^k(x)}{k!}(x)^k = f(0) + f'(a)(x) + \cdots + \frac{f^n(a)}{n!}(x)^n + \cdots$$

eg 88 Find the Maclaurin series of:

a) $f(x) = e^x$ Ans: $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ b) $f(x) = sin(x)$ Ans: $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$

c) $f(x) = cosh(x)$ Ans: $\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ d) $f(x) = ln(1-x)$ Ans: $-\sum_{k=1}^{\infty} \frac{x^k}{k}$

e) $f(x) = tan^{-1}(x)$ Ans: $\sum_{k=1}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$.

eg 89 Use a known Maclaurin series to find the series of :

a) $f(x) = sin(x)cos(x)$ Ans: $\frac{1}{2} \sum_{k=0}^{\infty} \frac{(2x)^{2k+1}}{(2k+1)!}$ b) $f(x) = e^{x^2}$ Ans: $\sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$

c) $f(x) = cos^2(x) = \frac{1+cos2x}{2}$ Ans: $\frac{1}{2} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!}$.

eg 90 Find the Taylor series of $f(x) = \frac{1}{x}$ at $x = 2$. Ans: $\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} (x-2)^k$.

eg 83 Use a known Taylor series to find the series of :

a) $f(x) = \frac{1}{x}$ about $x = 1$. Ans: $\frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{k=0}^{\infty} (-1)^k (x-1)^k$. $0 < x < 2$.

b) $f(x) = sin(x)$ about $x = \pi/2$. Ans: $sin(x) = cos\left(x - \frac{\pi}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(x - \frac{\pi}{2}\right)^{2k}$

eg 91 Find the sum of the following series

a) $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} = sin\left(\frac{\pi}{2}\right) = 1$ b) $\sum_{k=0}^{\infty} \frac{(ln(2))^k}{k!} = 2$.

eg 92 Use a series to find the limit of:

a) $\lim_{x \to 0} \frac{e^x - x - 1}{x^2} = \lim_{x \to 0} \frac{\sum_{k=0}^{\infty} \frac{x^k}{k!} - x - 1}{x^2} = \lim_{x \to 0} \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{x^2} = \lim_{x \to 0} \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots = \frac{1}{2}$

b) $\lim_{x \to 0} \frac{sin(x)}{x} = \lim_{x \to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}{x} = \lim_{x \to 0} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!} = \lim_{x \to 0} 1 + \frac{x^2}{3!} + \cdots = 1$.

eg 93 Find the Maclaurin series of $f(x) = \frac{e^{ix} + e^{-ix}}{2}$. Ans: $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$

Taylor Polynomials

The linearization of the function f at the point $x = a$ is the polynomial $p_1 = f(a) + f'(a)(x-a)$. If f has higher derivatives, it will higher order polynomial approximations for each derivative. Since these polynomials are obtained when the Taylor series is truncated, the n_{th} degree Taylor Polynomial for the function f at $x = a$ will be defined as $P_n(x) = \sum_{k=0}^{n} \frac{f^k(x)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^n(a)}{n!} (x-a)^n$ where $P_0(x) = f(a)$ is the zero degree Taylor Polynomial, or the zero degree approximation of f;

$P_1(x) = f(a) + f'(a)(x-a)$ is the first degree Taylor Polynomial, or the first degree approximation of f;
$P_2(x) = f(a) + f'(a)(x-a) + \frac{f'(a)}{2!}(x-a)^2$ is the second degree Taylor Polynomial, or the second degree approximation of f;
$\vdots$
$P_n(x) = f(a) + f'(a)(x-a) + \frac{f'(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$ is the n_{th}degree Taylor Polynomial, or the n_{th} degree approximation of f;

Remainder of a Taylor Polynomial

The error in approximating a function value $f(x)$ by a Taylor Polynomial will be given by its reminder after the n_{th} term $R_n(x)$. Since $f(x) = P_n(x) + R_n(x)$, the error of the approximation will be $|R_n(x)| = |f(x) - P_n(x)|$.

Taylor's Theorem
If f has $n+1$ derivatives in an open interval I containing a, then for each x in I, there is a number c between x and a such that
$f(x) = f(a) + f'(a)(x-a) + \frac{f'(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n(x)$, where the Lagrange's form of the reminder is given by $R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$.

Convergence of the Taylor Series
If $\lim_{n\to\infty} R_n(x) = 0$, Taylor Series converges to $f(x)$.. This is If $\lim_{n\to\infty} R_n(x) = 0$,
$f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^k(x)}{k!}(x-a)^k = \sum_{k=0}^{\infty} \frac{f^k(x)}{k!}(x-a)^k$ on the interval $|x-a| < R$ the radius of convergence of the series.

eg 94 Evaluate $\int_0^1 e^{-x^2}dx$ with an error < 0.001.

We can represent the integral as the alternating series
$\int_0^1 e^{-x^2}dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} dx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}\Big|_0^1 = x - \frac{x^3}{3.1!} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \frac{x^9}{9.4!} - \cdots\Big|_0^1 =$
$x - \frac{1}{3.1!} + \frac{1}{5.2!} - \frac{1}{7.3!} + \frac{1}{9.4!} - \cdots$. Trial and error show $R_{11} = \frac{x^{11}}{11.5!} \approx .0008 < .001$ so
$\int_0^1 e^{-x^2}dx = \sum_{k=0}^{4} \frac{(-1)^k}{k!\,(2k+1)} = .747487$ with an error $< .001$.

12.11 The Binomial Theorem

$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)(m-2)(m-n+1)}{n!}x^n + ...$

In closed form $y = 1+\sum_{k=1}^{\infty} \frac{m(m-1)(m-2)...(m-k+1)}{k!}x^k$. The binomial theorem can also be written as $\sum_{k=0}^{\infty} \binom{m}{k} x^k$, where $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ for $m \geq k$

eg 95 Show $\frac{m(m-1)(m-2)...(m-k+1)}{k!} = \binom{m}{k}$ for $m \geq 1$.

$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{m(m-1)(m-2)...(m-k+1)(m-k)(m-k-1)(m-k-2)\cdots(2)(1)}{k!\,(m-k)!} = \frac{m(m-1)(m-2)...(m-k+1)(m-k)!}{k!\,(m-k)!}$
$= \frac{m(m-1)(m-2)...(m-k+1)}{k!}$

eg 96 Show the interval of convergence of the Binomial series

$\rho = \lim_{n\to\infty} |\frac{m(m-1)(m-2)...(m-k+1)(m-k)x^{k+1}}{(k+1)!} \cdot \frac{k!}{m(m-1)(m-2)...(m-k+1)x^{k+1}}| =$
$\lim_{k\to\infty} |\frac{m-k}{(k+1)}x| = |x| < 1$, so the series converges to the function when $|x| < 1$.

Note: If m is a non negative integer, the series is finite and $(1+x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k$

eg 97 Find the series of $\frac{1}{(1+x)^2}$.

$\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} \binom{-2}{k} x^k =$
$1 + \frac{(-2)}{1!}x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \cdots + \frac{(-2)(-3)(-4)\cdots(-2-n+1)}{n!}x^n + \cdots =$
$1 - 2x + 3x^2 - 4x^3 + \cdots + (n+1)x^n + \cdots = \sum_{k=0}^{\infty} (-1)^k (k+1)\ x^k$

eg 98 Find the series of $\frac{1}{\sqrt{1-x}}$.

Since $\frac{1}{\sqrt{1-x}} = (1-x)^{-1/2}$, $\sum_{k=0}^{\infty} \binom{-1/2}{k} (-x)^k =$
$1 + \frac{(-\frac{1}{2})}{1!}(-x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(-x)^3 + \cdots$
$+ \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-k+1)}{k!}(-x)^k \cdots = 1 + \sum_{k=1}^{\infty} \frac{(1)(3)(5)\cdots(2k-1)}{2^k k!}x^k$.

eg 99 Find the series of $sin^{-1}x$.

Since $sin^{-1}x = \int \frac{1}{\sqrt{1-x^2}}dx = \int 1 + \sum_{k=1}^{\infty} \frac{(1)(3)(5)\cdots(2k-1)}{2^k k!}x^{2k}\,dx =$
$x + \sum_{k=1}^{\infty} \frac{(1)(3)(5)\cdots(2k-1)}{2^k k!}\,\frac{x^{2k+1}}{2k+1}$