9.1 Arc Length and The Mean Value Theorem (MVT)

MVT: If $f(x)$ is continuos in $[a, b]$ and differentiable on (a, b), $\exists$ a constant c, $a < c < b$, such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ or $\boldsymbol{f(b) - f(a) = f'(c)\,(b - a)}$.

Let $f(x)$ be a differentiable function in $[a, b]$. If f is subdivided in n parts in $[a, b]$, and if we take the limit as $n \to \infty$ of the sum of the length of all the parts, we obtain the length of the curve.

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} D(x_{i-1}, x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \overset{\text{MVT}}{=}$$
$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (f'(c)\Delta x)^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f')^2}\Delta x = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

If $f(x)$ is continuos in $[a, b]$ and differentiable on (a, b), the arc length formula is

$$L = \int_a^b \sqrt{1 + (dy/dx)^2}dx.$$

If $g(y)$ is continuos in $[c, d]$ and differentiable on (c, d), the arc length formula is

$$L = \int_c^d \sqrt{1 + (dx/dy)^2}dy.$$

eg 1 Find the Arc length of a circle of radius r. $Ans:\ 4\int_0^r \sqrt{\frac{r^2}{r^2-x^2}}dx.$

eg 2 Find length of the curve $y = x^2/2 - ln(x)/4$, $2 \le x \le 4$. $Ans:\ \int_2^4 (x + \frac{1}{4x})dx$

eg 3 Find length of the curve $y = ln(cos(x))$, $0 \le x \le \pi/3$. $Ans:\ \int_0^{\pi/3} sec(x)dx$

eg 4 Find length of the curve $y^2 = 2x$, $0 \le y \le 2$. $Ans:\ Ans:\ \int_0^2 \sqrt{1 + y^2}dy$

The Arc length Function is defined as $s(x) = \int_a^x \sqrt{1 + (f'(t))^2}dt$. This is a function that will give the arc length of a function from the point $(a, f(a))$ to $(x = f(x))$. If $x = b$, the function $s(b) = L = \int_a^b \sqrt{1 + (f'(x))^2}dx.$

If we differentiate $s(x)$, by the first part of the Fundamental Theorem of Calculus, we obtain $\frac{d}{dx}s(x) = \frac{d}{dx}\int_a^x \sqrt{1 + (dy/dt)^2}dt = \sqrt{1 + (dy/dx)^2}$. If we rewrite this equation, by multiplying by dx, we obtain the differential of arc length $ds = \sqrt{1 + (dy/dx)^2}dx$.

The Symmetric Form of the arc length formula $(ds^2) = (dx^2) + (dy^2)$ can be obtained by squaring both sides of the differential ds.

Centroids

The moment M is defined as $M = md$ where m is the mass and d is moment arm (distance).
The centroid (center of mass) is defined as:

For a discrete system of particles of mass m_i:
In one dimension, the moment about the origin is given by $M = \sum_i m_i\, x_i$,

so the center of mass will be $\overline{x} = \frac{M}{\sum_i m_i} = \frac{\sum_i m_i\, x_i}{\sum_i m_i}$.

In two dimensions, the moment about the axis is given by $M_x = \sum_i m_i\, y_i$ and $M_y = \sum_i m_i\, x_i$, so

the center of mass will be $\overline{x} = \frac{M_y}{m} = \frac{\sum_i m_i\, x_i}{\sum_i m_i}$ and $\overline{y} = \frac{M_x}{m} = \frac{\sum_i m_i\, y_i}{\sum_i m_i}$.

For a continuos one dimensional linear system with uniform (constant) mass density ρ (mass/unit length), the center of mass will be $\overline{x} = \frac{M_x}{m} = \frac{\int \rho x\, dx}{\int \rho dx} = \frac{\int x\, dx}{\int dx}$

for a two dimensional thin plate with mass density ρ (mass/unit length), the center of

mass will be $\overline{x} = \frac{M_y}{m} = \frac{\int_a^b \rho\, x\, [f(x) - g(x)]\, dx}{\int \rho [f(x) - g(x)] dx}$ and $\overline{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b \rho\, [f^2(x) - g^2(x)]\, dx}{\int \rho [f(x) - g(x)] dx}$.

eg 5 Find the centroid of the triangular lamina bounded by $y = 2x,\ y = 0, x = 1$ with uniform density $3 gm/cm^2$.

$\overline{x} = \frac{\int_0^1 3x\, [2x - 0]\, dx}{\int_0^1 3(2x) dx} = \frac{2}{3}$; and $\overline{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_0^1 3\, [(2x)^2 - 0^2]\ dx}{\int_0^1 3(2x) dx} = \frac{2}{3}$

10.1 Modeling with Differential Equations

A Differential Equation (DE) is an equation that contains derivatives or differentials.
The notation $\frac{\mathrm{d}\,y}{\mathrm{d}\,x}$ is called Leibniz notation where y is the dependent variable and x is the independent variable. In physical sciences and engineering Newton's dot notation is used $\ddot{y}$, where y is the dependent variable and t is the independent variable; y' is called prime notation where y is the dependent variable and x is the independent variable.
Consider the DE $\frac{d\,y}{d\,x} = 2x$.
Upon integration we obtain the solution (general solution) $y = x^2 + C$.
This solution will give a family of solution curves that can be plotted in the xy plane. A particular solution will be one of the curves of that family of solutions. That particular solution can be obtained by applying a condition (initial condition) to the DE.

eg 6 Find the particular solution of $\frac{dy}{dx} = 2x$ with condition $y(2) = 3$.
Since the general solution is $y = x^2 + C$, if we apply the condition $y(2) = 3$,
we obtain $3 = 2^2 + C$, $C = -1$, so the particular solution is $y = x^2 - 1$.
Note: A DE with initial conditions is called an Initial Value Problem (IVP), and the solution of an IVP gives particular solution of the family of solutions.

eg 7 Since a solution of a DE is a function that satisfies the DE,
a) Show that $y = e^x$ satisfies $y' - y = 0$. b) Show that $y = e^{-t}$ satisfies $\dot{y} + y = 0$
c) Show that $y = cos(x)$ satisfies $y'' + y = 0$. d) Show that $y = e^{-x}$ satisfies $y'' - y = 0$.
e) Show that $y = sinh(x)$ satisfies $\frac{d^2y}{dt^2} - y = 0 = 0$.

Some Models

1 A population where the growth is proportional to the population present at any time t can de described by the first order DE $\frac{dP}{dt} = kP$ with solution $P = P_0 e^{kt}$ where P_0 is the initial population.

2 A population growth in a limited environment is described by the Logistic Equation $\frac{dP}{dt} = kP(L - P)$ where L is the limiting population.

3 A mass spring system that satisfies Hook's Law can described by the second order DE $m\frac{d^2x}{dt^2} = -kx$ or $\frac{d^2x}{dt^2} = -\frac{k}{m}x$ with solutions $x = cos\left(\sqrt{\frac{k}{m}}t\right)$ or $x = sin\left(\sqrt{\frac{k}{m}}t\right)$.

A DE of the form $\frac{dy}{dx} = f(y)$ (like model 2) will always have equilibrium solutions.

eg 8 The DE $\frac{dP}{dt} = kP(200 - P)$ will have equilibrium solutions at $P = 0$ and $P = 200$.

10.3 Separation of Variables

Consider the first order ODE $\frac{dy}{dx} = f(x)g(y)$. If we make one side of the DE a function of one variable and the other side a function of the other variable, we have separated the variables, and a solution can be found by integration.

Models I and II can be solved by the method of separation of variables.

eg 9 Show that y is a solution of the differential equation.
a) $y = x^2 + 1$; $xy' = 2(y - 1)$ b) $y = 4x + 2$; $y' = (y - xy')^2$

eg 10 Solve the initial value problem:
a) $\frac{dy}{dx} = 2xy$, $y(0) = -3$; ANS: $y = -3e^{x^2}$
b) $\frac{dy}{dx} = 1 - y$, $y(0) = 2$; Ans: $y = 1 + e^{-x}$
c) $\frac{dy}{dx} = e^{(x+y)}$, $y(0) = 0$; Ans: $y = -ln(2 - e^x)$
d) $\frac{dy}{dx} = 4 + y^2$, $y(0) = \frac{2}{\sqrt{3}}$ Ans: $y = 2tan\left(2x + \frac{\pi}{6}\right)$

eg 11 Find a family of solutions of:
a) $\frac{dy}{dx} = \frac{y}{x^2 - x}$ Ans: $y = C\frac{x-1}{x}$
b) $\frac{dy}{dx} = \frac{2}{y\sqrt{x^2-1}}$ Ans: $y^2 = 4ln\left(x + \sqrt{x^2 - 1}\right)$
c) $\frac{dy}{dx} = y\, ln(x)$ Ans: $y = cx^x e^{-x}$

Applications:

eg 12 When a substance is dissolved in a liquid, the rate at which the weight (in pounds) of the dissolved substance $W(t)$ increases is given by the logistic equation $\frac{dW}{dt} = \frac{2}{3}W(3 - W)$. (a) Sketch the slope field of the equation, and discuss its solutions, (b) what is the maximum weight that can be dissolved in the liquid? Ans: $y = 3lb$

eg 13 Find the orthogonal trajectories of $y = e^{cx}$ as an implicit function.
Ans: $2y^2 ln(y) = -2x^2 + y^2 + C$

eg 14 A Mixture Problems
Rate of change of substance $Y(t)$, $\left(\frac{dY}{dt}\right) = rate\, in\left(\frac{\text{lb.}}{\text{min.}}\right) - rate\, out\left(\frac{\text{lb.}}{\text{min.}}\right)$.
At $t = 0$ a tank contains 25 lb. of salt dissolved in 50 gal. of water. Brine containing 4 lb./gal is allowed to enter at a rate of 2 gal/min. If the solution is drained at the same rate find the amount of salt as a function of time $Y(t)$. a) Find a function that gives the amount of salt at time t. b) Find the amount of salt in the tank after one hour. c) Find the concentration of salt at time t.
Ans: $\frac{dY}{dt} = 8 - \frac{y}{25} \Rightarrow Y(t) = 200 - 175\, e^{-\frac{t}{25}}$. Conc. $Y_c(t) = Y(t)/50 = 4 - \frac{7}{2}\, e^{-\frac{t}{25}}$
In all problems, (a) set the DE (b) solve the DE (c) answer the question(s).

10.4 Exponential Growth and Decay

A population where the growth is proportional to the population present at any time t can de described by the first order DE $\frac{dP}{dt} = kP$ with solution $P = P_0 e^{kt}$ where P_0 is the initial population.

eg 15 The population of a certain state is known to grow at a rate proportional to the number of people presently living in the state. If after 10 years the population has tripled and if after 20 years the population is 150,000, find the number of people initially living in the, state. Ans: $P_0 = 150,000/9 \approx 16666$ people.

eg 16 The relative rate of increase of the price of apples is 3 % per year. If a pound of apples cost \$0.40 at the beginning of 2000, what would they cost at the beginning of 2020. Ans: $0.4e^{0.6} \approx$ \$0.73

eg 17 The air pressure $P(y)$, in kg/m^2, at an altitude of $y\ km$ above the surface of the earth satisfies $\frac{dP}{dy} = -0.12p$. If the air pressure at the surface of the earth is $10^4 kg/m^2$,What is it at an altitude of $10\,km$. Ans: $P = 10^4 e^{-1.2}$

eg 18 A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there are 100 milligrams of the material present and if after two years it is observed that 5% of the original mass has decayed, find (a) an expression for the mass M at any time t and (b) the time necessary for 10% of the original mass to have decayed. Ans: $M(t) = 100\sqrt{.95}^{\,t}$; $t = \frac{2ln(.9)}{ln(.95)} yr. \approx 4.11 yr.$

eg 19 Use Newton's Law of Cooling $\frac{dT}{dt} = k(T_m - T)$ to solve the following problem.
A body at a temperature, of 50º F is placed in an oven whose temperature is kept at 150º F. If after 10 minutes, the temperature of the body is 75º F, (a) find the temperature of the body after 20 minutes, (b) find the time required for the body to reach a temperature of 100º F. Ans: $T(20) = 150 - 100(.75)^2$; $t = \frac{-10ln(2)}{ln(.75)} \approx 24.1 min.$

11.1 Parametric Equations

Curves that are not functions can be represented as a pair of functions of the form $x = f(t), y = g(t)$ where t is a parameter. Equations represented in this way are called parametric equations.

Cartesian Representation of Parametric Curves

One way of sketching parametric equations is by eliminating the parameter. This will give us the Cartesian representation of the curve. Parametric equations have orientation (direction of increasing parameter) and some times this orientation is lost when we eliminate the parameter t.

eg 20 Show that the parametric equation $x = a\,cos(t),\ y = a\,sin(t),\ 0 \le t \le 2\pi$ describes a circle or radius a.
We eliminate the parameter t by squaring x and y and adding.
So $x^2 = a^2 cos^2(t), y^2 = a^2 sin^2(t)$ or $x^2 + y^2 = a^2$ is an equation of a circle of radius a.

Parametric equations have orientation.
eg 21 Graph :
a) $x = cos(t),\ y = 4sin^2(t),\ \ 0 \le t < \pi$;
b) $x = \frac{t}{2},\ y = 4 - t^2,\ -2 \le t \le 2.$
a) If we eliminate the parameter t, $x^2 + \frac{y}{4} = 1$ or $y = 4 - 4x^2$. This is a segment of the parabola with vertex $(0,4)$ that opens down above the $x - axis$ with a counterclockwise orientation. $((1,0)$ for $t = 0$ and $(-1,0)$ for $t = \pi$).
b) If we eliminate the parameter t, $y = 4 - 4x^2$. This is the same parabola with a different parametrization oriented clockwise.
i.e. $((-1,0)$ for $t = -2$ and $(1,0)$ for $t = 2$).

eg 22 Graph $x = b\,cos(t),\ y = a\ sin(t),\ 0 \le t \le 2\pi\,, a \ge b$
This is an ellipse with a counterclockwise orientation with major axis of $2a$ on the $y - axis$.
eg 23 Graph $x = cosh(t),\ y = sinh(t),\ \ -3 < t < 3$. Check with your grapher.
$cosh^2(t) - \ sinh^2(t) = 1 \rightarrow\ x^2 - y^2 = 1$.
Since $x = cosh(t) \ge 1$, we have the right side of a unit hyperbola.
eg 24 Graph $x = \ -cosh(t),\ y = \ sinh(t),\ \ -3 < t < 3$. Check with your grapher.
Since $cosh^2(t) - \ sinh^2(t) = 1$ we have $x^2\ - y^2 = 1$. Since $cosh\ (t) > 0$, then $x < 0$. Since $sinh(t)\ < 0$ when $t < 0$ and $sinh(t) > 0$ when $t > 0$, then $y < 0$ for $-3 < t < 0$ and $y > 0$ for $0 < t < 3$. This is a segment of a hyperbola to the left of the y-$axis$.
eg 25 Graph $x = sec(t),\ y = \ tan(t)$, for $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $-\pi < t < \pi$. Check with your grapher.
eg. Graph $x = csc(t),\ y = \ cot(t)$, for $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $-\pi < t < \pi$.Check with your grapher.
eg 26 Eliminate the parameter to find the Cartesian equation. Indicate the direction of increasing t. Graph the equation by hand. Check your answer with your grapher.

a) $x = \sqrt{t},\ \ y = t^3;\ \ 0 \le t \le 4$
b) $x = 2\,cos(t),\ y = sin^2(t)\ ;\ \ -\pi \le t \le \pi$
d) $x = ln(t),\ y = 2ln(t)\,; \qquad 1 \le t \le e$
d) $x = 3e^{-t} - 2,\ y = 4e^{-t} - 1;\ \ \ t \ge 0$

Parametric Representation of Cartesian Curves
The parametric representation of Cartesian curves is not unique.
eg 27 Find a parametric representation for $y = x^2 - 2x + 1$ in [1,2].
$x = t,\ y = t^2 - 2t + 1;\ 1 \le t \le 2$ is one representation, and $x = t + 1,\ y = t^2;\ 0 \le t \le 1$ is another representation.

eg 28 A spider on the floor is crawling such that its position at any time t seconds is given by C_1: $x = 2t,\ y = 6t$ for $t \ge 0$. At the same time, a small bug has a position C_2: $x = t + 2,\ y = 2t^2 + 4$.
a) Will the two path intersect. b) Will the spider gets the bug? If so, at what point.
Check Graphically. (hint: Change the parameter of C_2 to u.If the system of equations in t and u has a solution, the two path intersect. If t and u are the same, the spider and the bug will meet).

eg 29 The path of two objects in the x-y plane are given parametrically by $C_1 : x = t + 1$, $y = t^2$, and $C_2 :\ x = t,\ y = 5 - 2t$
Find algebraically the intersection of the two path .

eg 30 The position of two objects in the x-y plane are given parametrically by $x = 3t,\ y = 2t^2$, and $x = 4t - 2,\ y = 4t$. Find algebraically if the two objects collide. If so, at what point.

11.2 Calculus with Parametric Curves

Sometimes it is difficult to eliminate the parameter to express y as a function of x. Nevertheless by the use of the chain rule we can find the derivative of $y = F(x)$ without eliminating the parameter.

To find $\frac{dy}{dx} = F'(x)$, let $y = F(x)$ with $x = f(t)$ and $y = g(t)$. If we substitute f and g into $y = F(x)$, we obtain $g(t) = F(f(t))$. By the chain rule

$\dot{g}(t) = F'(f(t))\dot{f}(t) = F'(x)\dot{f}(t)$ or $F'(x)=\frac{\dot{g}(t)}{\dot{f}(t)} = \frac{dy(t)}{dt} / \frac{dx(t)}{dt} = \frac{\dot{y}}{\dot{x}}$

where we have used the dot notation.

The second derivative is $\frac{d^2y}{dx^2} = \frac{d}{dx}F'(x) = \frac{\frac{d}{dt}\left(\frac{\dot{y}(t)}{\dot{x}(t)}\right)}{\frac{dx}{dt}}$.

We can find horizontal tangent lines at points where $\frac{dy(t)}{dt} = 0$ given that $\frac{dx(t)}{dt} \neq 0$.

We can find vertical tangent lines at points where $\frac{dx(t)}{dt} = 0$.

eg 31 Find the equation of the tangent line to $x = 2\,cos(t),\ y = 2sin(t)$ at $\left(1, \sqrt{3}\right)$.

eg 32 Find the equation of the tangent line to $x = t\,sin(t),\ y = t\,cos(t)$ at $t = \pi$

eg 33 Find the first and second derivatives of $x = sec(t)\ , y = tan(t)$ at $t = \frac{\pi}{3}$ with and without eliminating the parameter. ANS: $\left(y' = \frac{2}{\sqrt{3}},\ y'' = \frac{-\sqrt{3}}{9}\right)$.

eg 34 Given the cycloid $x = r(t - sin(t)),\ y = r(1 - cos(t))$ find the equation of the tangent line at $t = \frac{\pi}{6}$.

eg 35 Find the first and second derivatives of the cycloid defined parametrically by $x = 3(t - sin(t)),\ y = 3(1 - cos(t))$ when t $=\pi/6$.

eg 36 At what points on the curve $x = t^3 + 4t,\ y = 6t^2$ is the tangent parallel to the line $x = -7t,\ y = 12t - 5$.

eg 37 Given the cycloid $x = r(t - sin(t)),\ y = r(1 - cos(t))$ find the points where the curve has vertical and horizontal tangent lines.

eg 38 Find the value(s) of t where $x = 3cos(2t),\ y = 3sin(2t)$ has horizontal tangent lines.

Arc Length of Parametric Equations

If $y = f(x)$, the length of a curve in $[a, b]$ is given by $L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$.

If $x = g(y)$, the length of a curve in $[c, d]$ is given by $L = \int_c^d \sqrt{1 + (g'(y))^2}\,dy$.

If $y = y(t)$ and $x = x(t)$, the arc length of a curve in parametric form will be given by

$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2 dt}$.

eg 39 Find analytically the arc length of $x = \frac{t^2}{2}, y = \frac{t^3}{3}$ in the interval $0 \leq t \leq 1$.

Projectile Motion

If a projectile is fired at an angle α above the horizontal with no air resistance, the position after t seconds is given by $x = x_0 + (v_0 cos(\alpha))t$, $y = y_0 + (v_0 sin(\alpha))t - \frac{1}{2}gt^2$ where (x_0,y_0) is the initial position, $v_0$is the initial velocity and g is the acceleration of gravity ($32ft/s^2$; $9.8m/s^2$.

eg 40 A projectile is fired from the ground with an initial velocity of $98m/s$ at an angle of $45°$.

a) Find the parametric equations of the position of the projectile.

Ans: $< 49\sqrt{2}t, 49\sqrt{2}t - 4.9t^2 >$

b) Find the range of the projectile. Ans: $x(10\sqrt{2}) = 980m$

c) Find the maximum height of the projectile. Ans: $y\left(5\sqrt{2}\right) = 245$

d) Eliminate the parameter to show that the path is parabolic.

e) Find the vertex of the parabola and compare with part c Ans: (490,245)

f) Graph the path.

g) Find the x and y components of the velocity at the time of impact. Ans: $x = 49\sqrt{2}$, $y = -49\sqrt{2}$

h) Is this velocity the same as the initial velocity. Yes; $98m/s$.

eg 41 Do steps a-c in the previous example for a gun that is fired from a 20 $ft.$ high building with an initial velocity of 32 ft/s at an angle of $\frac{\pi}{6}$.

eg 42 Tiger hits a golf ball from the ground at an angle of 60° with respect to the ground and a velocity of 48 ft/s.

a) Determine the parametric equations that models the path of the ball.

b) Find how long the ball is in flight and how far will it gets.

c) Use calculus to find the maximum height of the ball.

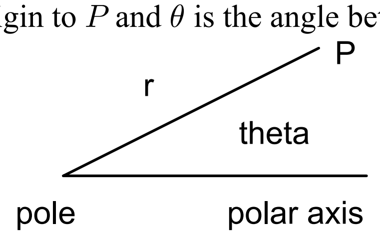
d) Determine the rectangular equation by eliminating the parameter.

e) With your answer in d, check the answers b and c.

11.3 Polar Coordinates

Polar Coordinate System

The polar coordinate system consists of a fixed point called pole (origin) and a ray called polar axis. a pair of polar coordinates (r, θ) can be associated with any point P where r is the line segment from the origin to P and θ is the angle between the polar axis and r.



Polar coordinates of points are not unique.

The point $(x, y) = (-1, 0)$ can be represented as $(r, \theta) = (1, \pi) = (1, -\pi) = (1, \pi \pm 2n\pi)$ $n = 1, 2, 3, ...$

To plot negative values of r, first locate θ and then plot r in the opposite direction of θ.

eg 43 To plot $(r, \theta) = \left(-1, \frac{\pi}{4}\right)$, first locate $\frac{\pi}{4}$ and plot one unit in the opposite direction of $\frac{\pi}{4}$. This is the same as plotting $\left(1, \frac{5\pi}{4}\right)$. Negative r is a shift of π in θ.
eg 44 give all the possible polar representations of the polar point $\left(2, \frac{\pi}{4}\right)$.
$\left(2, \frac{\pi}{4} \pm 2n\pi\right), \left(-2, \frac{5\pi}{4} \pm 2n\pi\right)\ n = 1, 2, 3, ...$

The transformation equations between polar and rectangular(Cartesian) coordinate system are
$x = r\,cos(\theta),\ y = r\,sin(\theta),\ r^2 = x^2 + y^2,\ tan(\theta) = \left(\frac{y}{x}\right)$.

Transformation of points

eg 45 Transform the following points from Cartesian to polar
$(-1, -1)$,
eg 46 Transform the following points from polar to Cartesian
$\left(2, \frac{\pi}{3}\right), \left(2, \frac{2\pi}{3}\right), \left(2, \frac{17\pi}{3}\right)$.

Transformation of curves

eg 47 Transform $x^2 + y^2 = 2x$ to polar coordinates.
eg 48 Transform $r = 2sin(\theta)$ and $r = csc(\theta)$ to Cartesian Coordinates.
eg 49 Do the points a) $\left(2, \frac{\pi}{2}\right)$; b) $(-1, 2\pi)$ lie on the curve $r = \frac{2}{1 - cos(\theta)}$
a) $2 = \frac{2}{1 - cos\left(\frac{\pi}{2}\right)}$ satisfies
b) $-1 = \frac{2}{1 - cos(2\pi)} = \frac{2}{0}$ is undefined, but $(-1, 2\pi) = (1, \pi)$ gives $1 = \frac{1}{2 - cos(\pi)}$ that is true.
Since a polar point is not unique, we need to evaluate all possible representations of a point.

Graphs in Polar Coordinates
1) Lines
$rsin(\theta) = a$ horizontal line
$rcos(\theta) = b$ vertical line
$(\theta) = \theta_0$ line through the origin with slope $m = tan(\theta)$
eg 50 Express the line $ax + by = c$ in polar form.
$r(a\,cos(\theta) + bsin(\theta)) = c$ general line

2) Limaċons are equations of the form $r = a \pm b\,cos(\theta), r = a \pm b\,sin(\theta)$
Consider the following cases:
a) If |a/b|=1 the curve is called a cardioid. $r = 2 \pm 2\,cos(\theta)$; $r = -1 \pm sin(\theta)$
b) If |a/b|<1 the curve is limacon with inner loop. $r = 1 \pm 2cos(\theta)$; $r = 2 \pm 3\,sin(\theta)$
c) If 1<|a/b|<2 the curve is a dimpled limacon. $r = 3 \pm 2\,cos(\theta)$; $r = -7 \pm 4\,sin(\theta)$
d) If |a/b| $\geq$ 2 the curve is a convex limacon. $r = 2 \pm cos(\theta)$; $r = 3 \pm sin(\theta)$

3) Lemniscate are equations of the form $r^2 = \pm a^2 cos(2\theta)$; $r^2 = \pm a^2 sin(2\theta)$.
4) Spirals are equations of the form $r = a\,\theta$

5) Roses are equations of the form $r = a\,cos(n\theta), r = asin(n\theta)$.
If n is odd the rose has n equally spaced petals. If n is even the rose has 2n equally spaced petals.
eg 51 Use your grapher to graph $r = 8\,sin(10\,\theta)$

6) Butterflies are equations of the form $r = 1 + sin(n\theta) + cos^2(2n\theta)$
eg 52 Use your grapher to graph for $n = 2, 3, 4$

Symmetry in polar coordinates
If $\theta \to -\theta$ does not alter the equation, then the graph is symmetric w.r.t. the x-axis.
eg 53 $r = 1 + cos(\theta) = 1 + cos(-\theta)$ a cardioid
If $\theta \to \pi - \theta$ does not alter the equation, then the graph is symmetric w.r.t. the y-axis.
eg 54 $r = sin(\theta) = sin(\pi - \theta)$ a rose
If $r \to -r$ does not alter the equation then the graph is symmetric w.r.t. the origin.
eg 55 $r^2 = cos(2\theta)$ a lemniscate.
eg 56 Sketch $r = 2 - 4\,sin(\theta)$ in polar coordinates. Label the x and y intercepts.
eg 57 Sketch $r = csc^2(\theta/2)$ by transforming to rectangular coordinates. Label the x and y intercepts.

Tangents to Polar Curves
If $r = f(\theta)$ is a polar curve, we can obtain parametric equations by substituting r into $x = rcos(\theta)$, $y = rsin(\theta)$. This gives us the parametric equations $x = f(\theta)cos(\theta)$, $y = f(\theta)sin(\theta)$ with parameter θ.

Using the method for parametric curves, we have

$$\frac{dy}{dx} = \frac{dy}{d\theta}/\frac{dx}{d\theta} = \left(\frac{dr}{d\theta}sin(\theta) + rcos(\theta)\right)/\left(\frac{dr}{d\theta}cos(\theta) - rsin(\theta)\right)$$

eg. Find the slope of the tangent line for the cardioid $r = 1 - cos(\theta)$ at $\theta = \pi/4$.
Since $x = (1 - cos(\theta))cos(\theta)$, $y = (1 - cos(\theta))sin(\theta)$, $\frac{dy}{dx} = \frac{dy}{d\theta}/\frac{dx}{d\theta} =$
$\left(sin^2(\theta) + (1 - cos(\theta))cos(\theta)\right)/(sin(\theta)cos(\theta) - (1 - cos(\theta))sin(\theta))\,|_{\pi/4} = 1 + \sqrt{2}$

eg 58 Use the grapher. to sketch the graph of $r = 1 - cos(\theta)$ $0 \leq \theta \leq 2\pi$ in parametric equations.

11.4 Areas and Lengths in Polar Coordinates

An infinitesimal sector of a curve can be approximate by an infinitesimal sector of a circle. So $\frac{\Delta A_{sector}}{A_{circle}} = \frac{\Delta\theta}{2\pi}$, or $\Delta A_{sector} = \frac{1}{2}r^2\Delta\theta$. The area of a sector between the angles α and β is then given by the sum of the areas of all the infinitesimal sectors in that sector or $\int_\alpha^\beta = \frac{1}{2}r^2 d\theta$.

To find areas first sketch the region. To simplify computations look for symmetry. If the curves intersect, find the angles where the curves intersect to find the limits of integration.

eg 59 Find the area inside the curve $r = 2\ \sin(\theta)$.
eg 60 Find the area inside the rose $r = cos(2\theta)$.
eg 61 Find the area of the inner loop of $r = 2 + 4sin(\theta)$.
eg 62 Find the area inside the larger loop and outside the of the smaller loop of $r = 1 + 2sin(\theta)$
eg 63 Find the area of the region that is inside $r = 2sin\,(\theta\,)$ and outside $r = \sqrt{3}$.

Arc Length of Polar Curves

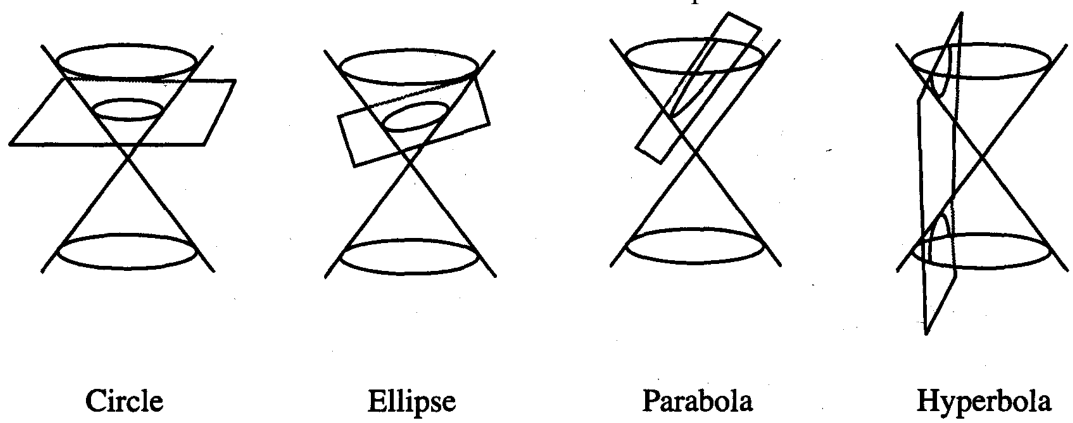
If a curve has polar equation $r = f(\theta)$, where is continuous for $\alpha \le \theta \le \beta$ and $\frac{dr}{d\theta} = f\,'(\theta)$ is differentiable in $\alpha < \theta < \beta$, then its arc length L from α to β is given by
$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$

Proof:
Since $r = f(\theta),\ x = rcos(\theta) = f(\theta)cos(\theta),\ y = rsin(\theta) = f(\theta)sin(\theta)$.
Also $\frac{dx}{d\theta} = -f(\theta)sin(\theta) + f'(\theta)cos(\theta);\ \frac{dy}{d\theta} = f(\theta)cos(\theta) + f'(\theta)sin(\theta)$ and
$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f\,'(\theta)]^2$.
So we can say $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f\,'(\theta)]^2} d\theta =$
$\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$

eg 64 Find the arc length by using the formula for arc length in polar coordinates of:
a) the circle of radius 2 b) the spiral $r = e^\theta\ 0 \le \theta \le 1$
c) the cardioid $r = 1 + cos(\theta)$ d) the spiral $r = \theta$.

11.5 Conic Sections

The standard conic sections are the curves in which a plane cuts a double cone.



Parabolas

A parabola is the locus of points in the plane that are equidistant from a fixed point F. (called the focus) to a fixed line l (called the directrix).

If the point $P(x, y)$ is on the parabola the point $F(0, p)$ is the focus and $y = -p$ is the equation of the parabola, the equation of the parabola becomes $x^2 = 4py$.

If $p > 0$, the parabola opens up, and if $p < 0$, the parabola opens down.

If the point $P(x, y)$ is on the parabola the point $F(p, 0)$ is the focus and $x = -p$ is the equation of the parabola, the equation of the parabola becomes $y^2 = 4px$.

If $p > 0$, the parabola opens to the right, and if $p < 0$, the parabola opens to the left.

The focal diameter of a parabola has a length of $4p$.

eg 65 Describe the following parabolas with vertex at the origin:

a) $y = 5x^2$ Ans: F:(0,/1/20)

b) Focus $F(5, 0)$ Ans: $y = x/20$

c) Focal diameter 8 and focus on the negative y-axis. Ans: $x^2 = -2y$

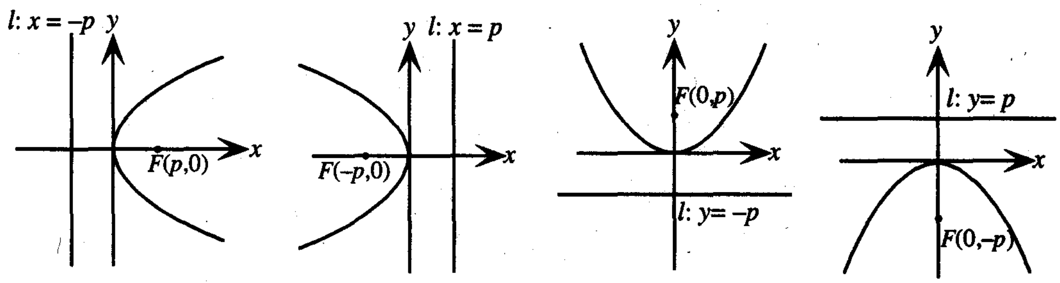
d) Through $(4, -2)$ and opening to positive x-axis. Ans: $y^2 = x$

e) Find the length of the parabolic arc intercepted by the focal diameter of $x^2 = 4py$ for $p = 1/2$.

For the general form, $L = 2\int_0^{2p}\sqrt{1+(y')^2}dx = 2\int_0^{2p}\sqrt{1+\left(\frac{x}{2p}\right)^2}dx = 4p\int_0^{\pi/4}sec^3\theta d\theta = 2p[\sqrt{2}+ln\left(1+\sqrt{2}\right)]$.

So for $p = 1/12$, the Parabolic arc of $x^2 = 2y$ is $\sqrt{2}+ln\left(1+\sqrt{2}\right)$

$y^2 = 4px\ p > 0$ $\quad y^2 = 4px\ p < 0$ $\quad x^2 = 4py\ p > 0$ $\quad x^2 = 4py\ p < 0$



Shifted Parabolas

If the vertex of a parabola is shifted to the point (h, k), the equations of the parabola becomes $(x-h)^2 = 4p(y-k)$ or $y = ax^2 + bx + c$ if the parabola opens to y. The equations of the parabola becomes $(y-k)^2 = 4p(x-h)$ or $x = ay^2 + by + c$ if the parabola opens to x.

eg 66 Graph the parabola $y = x^2 - 4x + 2$

eg. Find the equation of the parabola, with axis of symmetry $y = 0$, through the points (3,2) and (2,-3),and Graph. Ans: $y^2 = -5(x - 19/5)$

eg 67 Find the equation of the parabola, with the axis of symmetry parallel to the y-axis, through the points $(1,2)$,$(2,9)$ and $(3,22)$.

eg 68 Find the equation of the parabola, with the axis of symmetry parallel to the x-axis, vertex $(5,-10)$ and $y_{int} = -5$.

eg 69 Find the equation of the parabola, with the axis of symmetry parallel to the x-axis, intercepts $(,0,-4)$,(0,2), through the point $(8,-2)$. Ans: $(y+1)^2 = -(x+9)$

Ellipses

An ellipse is the locus of points in the plane whose sum of distances from two fixed points $F_1$and F_2(called the foci) is a constant.

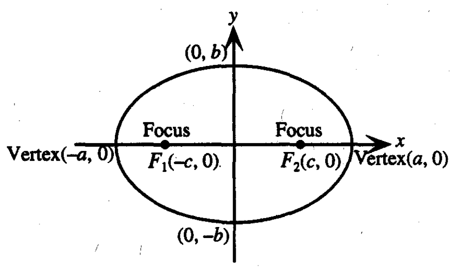
Let $P(x, y)$ be point on the ellipse, $F(\pm p, 0)$ be the foci, $V(\pm a, 0)$ be the vertices and the sum of the distances be $2a$. So we have $d(P, F_1) + d(P, F_1) = 2a$; or $\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$. After simplification, the equation of the ellipse becomes $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$ or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b^2 = a^2 - c^2 (a > b > 0)$. This ellipse has a horizontal major axis of length $2a$ and a vertical minor axis of length $2b$.

If we interchange $x \Leftrightarrow y$, The ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$will have a vertical major axis of length $2a$ and a horizontal minor axis of length $2b(a > b > 0)$.

Ellipse with foci on x-axis

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

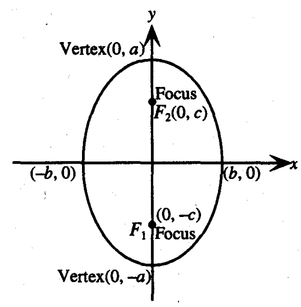
$$a>b>0$$



Ellipse with foci on y-axis

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

$$a>b>0$$



The eccentricity of an ellipse is a measure of the deviation of an ellipse from being circular by the ratio c/a . So $e\ =\frac{c}{a}$ with $c=\sqrt{a^2-b^2}$ and $0<e<1$,
with $e=0$ (no deviation) being a circle.

eg 70 Describe the following ellipses with center at the origin:
a) $4x^2+9y^2=36$
b) $F:(0,\pm 3);\ V:(0,\pm 5)$
c)$F:(8,0), e=4/5$
d) Major axis 6, Minor axis 4, foci on the x-axis.
eg 711 Consider a particle traveling clockwise on the elliptical path $\frac{x^2}{100}+\frac{y^2}{25}=1$. If the particle leaves the orbit at the point $(-8,3)$ and travels in a straight line tangent to the ellipse, find the point at which the particle crosses the y-axis.
eg 72 Find the equation of the ellipse with foci in the x-axis with an inscribed circle of radius 10 tangent to the ellipse such that the area of the ellipse is twice the area of the circle. The circle and the ellipse have the same center.
eg 73 Find the volume of the solid generated when area bounded by $\frac{x^2}{3}+\frac{y^2}{1}=1, y=x$ and $x=0$ in the first quadrant is rotated about the y-axis.

Shifted Ellipses

If the center of an ellipse is shifted to the point (h,k), the equations of the ellipse becomes $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.
eg 74 Find the equation of the conic with foci at $(\pm 1,2)$ length of minor axis 6. Graph.
eg 75 Graph the conic $9x^2+4y^2-18x+8y-23=0$.
eg 76 Find the locus of points (x,y) such that the sum of their distances from the fixed points $(1,1)$ and (1,3) is $2\sqrt{2}$. Graph.

Hyperbolas

A hyperbola is the locus of points in the plane whose difference of distances from two fixed points $F_1$and F_2(called the foci) is a constant .

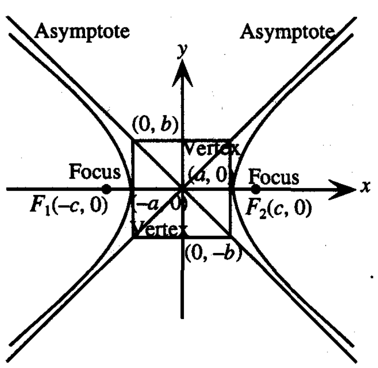
Let $P(x, y)$ be point on the hyperbola, $F(\pm p, 0)$ be the foci, $V(\pm a, 0)$ be the vertices and the difference of the distances be $\pm 2a$. So we have

$d(P, F_1) - d(P, F_1) = \pm 2a$; or $\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a$. After simplification, the equation of the hyperbola becomes $\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$ or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b^2 = c^2 - a^2 (a > 0; b > 0)$. This hyperbola has a horizontal transverse axis of length $2a$ and a vertical conjugate axis of length $2b$, with asymptotes $y = \pm \frac{b}{a}x$.

If we interchange $x \Leftrightarrow y$, The hyperbola $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ will have a vertical transverse axis of length $2a$ and a horizontal conjugate axis of length $2b$, with asymptotes $y = \pm \frac{a}{b}x$
$(a > 0; b > 0)$

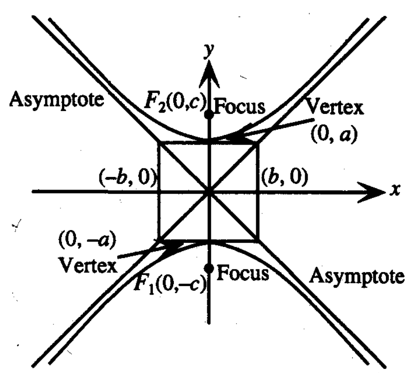
Hyperbola with foci on x-axis

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$a > 0; b > 0$$



Hyperbola with foci on y-axis

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$$
$$a > 0; b > 0$$



The eccentricity of a hyperbola is a measure of its shape the ratio c/a . So $e = \frac{c}{a}$ with $c = \sqrt{a^2 + b^2}$ and $e > 1$. When $e \approx 1$ we have a very narrow hyperbola.

eg 77 Describe the following hyperbolas with center at the origin:
a) $4x^2 - 9y^2 = -36$ b) $F(0, \pm 2)$; $V(0, \pm 1)$ c) $a = 2, e = 5/4$.
d) $y = \pm \frac{3}{2}x$ and transverse axis on x of length 8. e) Foci $(0, \pm 8)$, asymptotes $y = \pm \frac{1}{2}x$.

eg 78 Find the volume of the solid generated when the area between $y^2 - x^2 = 1$, $0 \le x \le 1$ is rotated about the x-axis.

Shifted Hyperbolas

If the center of an ellipse is shifted to the point (h, k), the equations of the ellipse becomes $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$

eg 79 Describe the conic $x^2 - y^2 = 10(x - y) + 1.$

eg 80 Describe the conic $9x^2 - y^2 + 18x + 6y = 9.$

eg 81 Describe the conic with foci (1,5) and (1,7) and eccentricity $e = 2.$

eg 82 Find the locus of points such that the absolute value of the difference of their distances from $(3, 6)$ and (3,0) is 4. Graph.

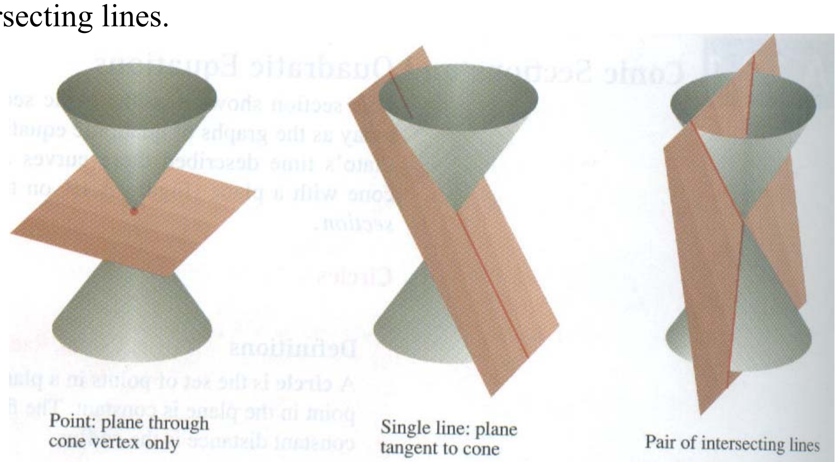
General Form of the Equation of Conics

$$\boldsymbol{Ax^2 + By^2 + Cx + Dy + E = 0}$$

1) If $A = B = 0$, we have lines; If $A = B = C = 0$; we have horizontal lines; If $A = B = \mathrm{D} = 0$; we have vertical lines; If $A = B = C = D = 0$, we have the origin.
2) If A or B are zero, we have parabolas.
3) If $A = B \neq 0$, we have circles.
4) If $A \neq B$ same signs, we have ellipses.
5) If sign$A \neq$ sign B, we have ellipses.

Degenerate Conics

We have degenerated conics when the equation of the conic is a single point, a single line or a pair of intersecting lines.



eg 83 Sketch graphs of :

a) $9x^2 - y^2 + 18x + 6y = 0$

b) $4x^2 + y^2 - 8x + 2y + 6 = 0$