7.2 Exponential Functions and Their Derivatives

An exponential function is a function with its independent variable in an exponent. The basic exponential function is defined as $f(x) = ab^x$, b >0, $b \neq 1$. a is the y intercept, and b is called the base. If $0 < b < 1$, the function describes decay. If $b > 0$ the function describes growth.

Some Terminology:
Let i be an initial number and f a final number;
[Growth factor from i to f] $= \frac{f}{i}$
[Relative change from i to f] $= \frac{f-i}{i}$
We can see that the relative change from i to f is $\frac{f-i}{i} = \frac{f}{i} - 1 =$ growth factor $- 1$ or the growth factor $= 1 +$ relative change .

eg 1 Consider a population of 100 that increases by a factor of 3 every month so $P(\text{t}) = 100 \times 3^t$ where t is given in month. (growth factor 3 per month)

eg 2 Consider a population of 100 that triples every 2 years, so $P(\text{t}) = 100 \times 3^{t/2}$ where t is given in years, (growth factor of $3^{1/2}$per year), or $P(\text{t}) = 100 \times 3^{t/24}$ where t is given in months. (growth factor of $3^{1/24}$per month)

eg 3 Consider a population of 100 increases by 2% every 3 years, so $P(\text{t}) = 100 \times (1 + .02)^{t/3}$ where t is given in years, with growth factor of (1.02) every 3 years or growth factor of $(1.02)^{1/3}$ per year with relative change of 2% every 3 years.

eg 4 Consider a population of 100 decreases by $1/5$ every 1/2 year, so $P(\text{t}) = 100 \times (1 - .2)^{2t}$ where t is given in years or $P(\text{t}) = 100 \times (1 - .2)^{t/6}$ where t is given in months.

eg 5 Graph $f(x) = 2^x$; $f(x) = 3^x$; $f(x) = e^x$; $f(x) = 2^{x-1} + 1$; $f(x) = -e^{-x+1}$

Compound Interest :
Suppose you invest \$ 100 (the principal P) at an interest rate of $r = 10\%$ reinvested at the end of the year. The table below shows the amount of money A you will have after t years.

t	0	1	2	3	t
$A(t)$	100	110	121	133.10	
$A(t)$	100	$100(1+.1)$	$100(1+.1)^2$	$100(1+.1)^3$	$100(1+.1)^t$

In general you will have the compound interest formula $A(t) = P(1+r)^t$.

If you compound n times per year, the formula becomes $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$.
The table below shows the same investment after one year for different values of n.

	$Annually$	$semiannually$	$Quarterly$	$Monthly$	$Daily$
n	1	2	4	12	365
$A(1)$	110	110.25	110.38	110.47	110.52

We can see that as n increases, the rate of increase is decreasing.

To consider what happens as $n \to \infty$ lets find $\lim\limits_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}$.
Let $\frac{r}{n} = \frac{1}{x}$ or $n = xr$.Since as $n \to \infty$, $x \to \infty$, the limit becomes
$\lim\limits_{x\to\infty} P\left(1+\frac{1}{x}\right)^{xrt} = P\left[\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{rt}$. Since it can be shown that
$\lim\limits_{x\to\infty}(1+x)^{x} = e$, the compound interest formula $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ becomes $A(t) = P\mathrm{e}^{\mathrm{r}t}$ as $n \to \infty$

eg 6 Show $e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = \lim\limits_{n\to 0}(1+n)^{1/n}$; $\lim\limits_{x\to-\infty} e^{x} = 0$; $\lim\limits_{x\to\infty} e^{x} = \infty$

eg 7 Find $\lim\limits_{x\to\frac{\pi}{2}^{+}} e^{tan(x)}$; $\lim\limits_{x\to 2^{-}} e^{1/(2-x)}$

Derivative of $f(x) = e^{x}$

By definition $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\frac{e^{(x+h)}-e^{x}}{h} = \lim\limits_{h\to 0} e^{x}\left(\frac{e^{h}-1}{h}\right) = e^{x}\lim\limits_{h\to 0}\left(\frac{e^{h}-1}{h}\right)$.
Since $\lim\limits_{h\to 0}\left(\frac{e^{h}-1}{h}\right) = 1$, if $f(x) = e^{x}$, $f'(x) = e^{x}$ where the slope of the tangent line at $x = x_0$ is the value of the function at that point .
If $f(x) = e^{u(x)}$, by the chain rule $f'(x) = e^{u(x)}\,\frac{du(x)}{dx}$.

eg 8 Find the derivative of: $f(x) = \boldsymbol{e^{tan\sqrt{x}}}$; $w(t) = \frac{e^{t}}{t^2-1}$; $s(t) = \frac{e^{\sqrt{t}}}{\mathrm{e}^{t}}$.

eg 9 Give the equation of the tangent line to $y = xe^{-x}$ at its inflection point.

Integrals Involving Exponential Functions

eg 10 Evaluate $\int_0^1 sec^2(x)e^{(tan(x))}dx$; $\int e^{1/x}/x^2 dx$; $\int \mathrm{e}^{ax}\,dx = \frac{1}{\mathrm{a}}\mathrm{e}^{ax} + c$

eg 11 Find the average value of $f(x) = 2xe^{-x^2}$ in $[0,3]$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{3}\int_0^3 2xe^{-x^2}dx = \frac{4-e^{-9}}{3}$$

7.3 Logarithmic Functions

Logarithmic function as the inverse of the exponential function

Consider $f(x) = 2^x$

1) 1-1 by graph

2) $x = 2^y$; $y = log_2 x = f^{-1}(x)$

3) $f\left(f^{-1}(x)\right) = f^{-1}(f(x)) = x$

The inverse $y = b^x$ is $y = log_b x$ where $log_b x = \begin{cases} < 0 & if\, 0 < x < 1 \\ 0 & if\, x = 1 \\ > 0 & if\, x > 1 \end{cases}$.

If $b = e$, the inverse of $y = e^x$ is $y = lnx$

Properties of logs:

1) $log_b b^x = b^{log_b x} = x$; (property of inverses)

2) $log_b(x*y) = log_b x + log_b y$; $log_b(x/y) = log_b x - log_b y$; $log_b x^y = y\, log_b x$

3) $log_c a = \frac{log_b a}{log_b c}$, $b^x = e^{ln\, b^x} = e^{(lnb)x}$ (change of base formulas)

Consider a population of base 2 given by $P(\mathrm{t}) = 100 \times 2^{t/3}$.
If we need to express the population in term of base e, $P(\mathrm{t}) = 100e^{kt}$, we have to change from base 2 to base e. So $2^{1/3} = e^k$ or $k = ln(2)/3$. k is called the **relative *rate* of change** of the population or $P(\mathrm{t}) = 100e^{ln(2)t/3}$.

The **percentage *rate* of change** = **relative *rate* of change × 100**

This rate will also be represented as $\frac{\dot{P}(\mathrm{t})}{P(\mathrm{t})} = k$.

eg 12 A man has a concentration of 10 mg/l of lead in his blood. His body eliminate the lead with half life time of 16 days. What will be the relative rate of change of the concentration of lead in his blood ?

Sol: Since we only know the derivative of an exponential with base e, we can change to base e, or $C(t) = 10 \times (1/2)^{t/16} = 10e^{ln(1/2)t/16}$. So $k = ln(1/2)/16 = -ln(2)/16$.

Derivative of $y = b^x$:

Let $f(x) = b^x$ $\quad \frac{d}{dx}(b^x) = \lim_{h \to 0} \frac{b^{(x+h)} - b^x}{h} = \lim_{h \to 0} b^x \left(\frac{b^h - 1}{h}\right) = b^x \lim_{h \to 0}\left(\frac{b^h - 1}{h}\right)$

But since $b^x = e^{(lnb)x}$, $\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{(lnb)x}\right) = e^{(lnb)x} ln(b) = ln(b) b^x$,

we have to assume $\lim_{h \to 0}\left(\frac{b^h - 1}{h}\right) = ln(b)$ since $\lim_{h \to 0} b^x \left(\frac{b^h - 1}{h}\right) = b^x ln(b)$.

If $f(x) = b^x$, $f'(x) = b^x ln(b)$

In general $\frac{d}{dx}\left(b^{u(x)}\right) = ln(b) b^{u(x)} \ \frac{du(x)}{dx}$.

Likewise $\int b^x \, dx = \int e^{x \, ln(b)} dx = \frac{b^x}{ln\,(b)} + c$

eg 13 Evaluate $\int 2^x + x^2 \, dx = \frac{2^x}{ln2} + \frac{x^3}{3} + c$.

eg 14 Evaluate $\int x 2^{x^2} \, dx = \frac{2^{x^2}}{2ln2} + c$.

eg 15 A man has a concentration of 10 mg/l of lead in his blood. His body eliminate the lead with half life time of 16 days. What will be the concentration of lead in his blood and how rapidly is it decaying after 32 days.

Sol: $C(t) = 10(1/2)^{t/16}$so $C(32) = \frac{5}{2} mg/l$. $\dot{C}(t) = 10(1/2)^{t/16} \frac{(-ln(2))}{16}$ so

$\dot{C}(32) = \frac{-5 \, l \, n(2)}{32} \frac{mg}{l \ day} \approx -10.8\% \ \frac{mg}{l \ day}$

eg 16 A population of 100 increases by 2% quarterly. Find the relative rate of change (a) per month (b) per year.

Sol: a) Quarterly (3 months) $\frac{\dot{P}(t)}{P(t)} = \frac{100(1.02)^{t/3}}{100(1.02)^{t/3}} \frac{ln(1.02)}{3} = \frac{ln(1.02)}{3}$.

If we change the base to e, $\quad 100(1.02)^{t/3} = 1004 e^{ln(1.02)^{t/3}} = 1004 e^{\frac{ln(1.02)}{3} t}$. Since r is the relative rate of change, $r = \frac{\dot{P}(t)}{P(t)} = \frac{ln(1.02)}{3}$.

b) Quarterly (4 times a year) $\frac{\dot{P}(t)}{P(t)} = \frac{100 \times (1.02)^{4t}}{100 \times (1.02)^{4t}} 4ln(1.02) = 4ln(1.02)$.

If we change the base to e, $\quad 100(1.02)^{4t} = 100 \times e^{ln(1.02)^{4t}} = 100 \times e^{4ln(1.02)t}$. Since r is the relative rate of change, $r = \frac{\dot{P}(t)}{P(t)} = 4ln(1.02)$.

7.4 Derivatives of Logarithmic Functions.

Let $f(x) = ln(x)$. If $y = ln(x)$, $e^y = x$ or $e^y \frac{dy}{dx} = 1$ so $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$.

In general $\frac{d}{dx} ln(x) = \frac{1}{x}$ or if $u = u(x)$, $\frac{d}{dx}(ln(u(x))) = \frac{1}{u(x)} \frac{du(x)}{dx}$.

eg 17 Find the derivative of :
$f(x) = ln\left(x^{2/3}\right)$; $g(x) = 2^{x^2} ln(x+1)$; $h(x) = sec(x)\, ln\sqrt{1-x^2}$

eg 18 Find $\frac{d}{dx} log_b(x)$.

Since $log_b(x) = \frac{ln(x)}{ln(b)}$, $\frac{d}{dx} \frac{ln(x)}{ln(b)} = \frac{1}{xln(b)}$.

In general $\frac{d}{dx}(log(u(x))) = \frac{1}{u(x)\ln(b)} \frac{du(x)}{dx}$.

eg 19 Find the derivative of $\frac{d}{dx} log_{10}\left(sin\left(e^{x^2}\right)\right)$

eg 20 Based on the formula D = 20 $\log_{10}$(500P) for the decibel level of a sound in terms of the variation P in air pressure caused by the sound, how rapidly is the decibel level of a scream increasing when the variation in air pressure from the scream is 10^{-4} pounds per square inch and is increasing 10^{-3} pounds per square inch per second? Ans: 200/ln10 db/s.

Logarithmic Differentiation

eg 21 Find $\frac{d}{dx}(x^{\sqrt{x}})$.

a) By change of base: $\frac{d}{dx}(x^{\sqrt{x}}) = \frac{d}{dx}(e^{\sqrt{x}\, ln(x)}) = x^{\sqrt{x}}\,(\frac{2+ln(x)}{2\sqrt{x}})$

b) By logarithmic differentiation: let $y = x^{\sqrt{x}}$, so $ln(y) = \sqrt{x}\, ln(x) \Rightarrow \frac{1}{y} y' = \frac{2+ln(x)}{2\sqrt{x}}$ or
$y' = x^{\sqrt{x}}\,(\frac{2+ln(x)}{2\sqrt{x}})$

eg 22 Find the derivative by logarithmic differentiation of :

$y = \frac{d}{dx}(x^n)$; $y = \sqrt{x}\, e^{x^2}\left(x^2+1\right)$; $y = \sqrt[4]{\frac{x^2+1}{x^2-1}}$ find $y'(x)$

Integrals Involving Logarithmic Functions

We know that $\frac{d}{dx}ln(x) = \frac{1}{x}$, but $\int \frac{1}{x}dx$ can only be $ln(x) + c$ for $x > 0$. To consider the antiderivative of $\frac{1}{x}$ for $x < 0$, we need first to consider $\frac{d}{dx}ln(-x)$ for $x < 0$. Since $\frac{d}{dx}ln(-x) = \frac{1}{x}$ is the same as $\frac{d}{dx}ln(x)$, we can combine the two of them as

$$\frac{d}{dx}ln|x| = \begin{cases} \frac{d}{dx}ln(x) & \text{for } x > 0 \\ \frac{d}{dx}ln(-x) & \text{for } x < 0 \end{cases} = \frac{1}{x} \text{ for all } x \neq 0.$$

With this result we can say $\int \frac{1}{x}dx = ln|x| + c$.

eg 23 Interpret $\int_{-5}^{-3} \frac{1}{x}dx = ln\left(\frac{3}{5}\right) < 0$ graphically.

eg 24 Evaluate $\int 2x\, cot\left(x^2\right)dx = ln(sin\left(x^2\right)) + c$

eg 25 Evaluate $\int_0^{\pi/3} sec(x)dx = ln\left(2 + \sqrt{3}\right)$

7.5 Trigonometric and Inverse Trigonometric Functions and their Derivatives

Review derivatives of Trigonometric functions:

eg 26 A light house is 4 miles from a pier on a line perpendicular to the straight shore. Let s be the distance from the pier to the place where the light hits the shore and θ the angle between the perpendicular to the shore and the light beam. If the light makes one revolution in 3 seconds, How fast is the beam of light moving along the shore when $\theta = 30°$. Solution; $s = 4tan\theta$ $\rightarrow \frac{ds}{dt} = 4sec^2(\theta)\frac{d\theta}{dt}|_{\theta=\pi/6} = 4\left(\frac{4}{3}\right)\frac{2\pi}{3} = \frac{32\pi}{9}mi/s.$

Inverse Trigonometric Functions

If we restrict $f(x) = sin(x)$ from $-\pi/2 \leq x \leq \pi/2$, then
$f^{-1}(x) = sin^{-1}(x) = arcsin(x)$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.
$sin^{-1}(sin(x)) = x\,,\ -\pi/2 \leq x \leq \pi/2$ and $sin\left(sin^{-1}(x)\right) = x\,,\ 1 \leq x \leq 1$

If we restrict $f(x) = cos(x)$ from $0 \leq x \leq \pi$, then $f^{-1}(x) = cos^{-1}(x) = arccos(x)$ with domain $[-1, 1]$ and range $[0, \pi]$.
$cos^{-1}(cos(x)) = x\,,\ 0 \leq x \leq \pi$ and $cos\left(cos^{-1}(x)\right) = x\,,\ 1 \leq x \leq 1$

If we restrict $f(x) = tan(x)$ from $-\pi/2 < x < \pi/2$, then $f^{-1}(x) = tan^{-1}(x) = arctan(x)$ with domain $(-\infty, \infty)$ and range $(-\pi/2, \pi/2)$.
$tan^{-1}(tan(x)) = x\,,\ -\pi/2 \leq x \leq \pi/2$ and $tan\left(tan^{-1}(x)\right) = x$, all x.

If we restrict $f(x) = cot(x)$ from $0 < x < \pi$, then $f^{-1}(x) = cot^{-1}(x) = arccot(x)$
$= \pi/2 - tan^{-1}(x)$

If we restrict $f(x) = sec\,(x)$ from $0 \leq x \leq \pi,\ x \neq \pi/2$, then $f^{-1}(x) = sec^{-1}(x) = arcsec(x)$
$= cos^{-1}(1/x)$

If we restrict $f(x) = csc(x)$ from $-\pi/2 \leq x \leq \pi/2$, $x \neq 0$, then
$f^{-1}(x) = csc^{-1}(x) = arccsc(x) = sin^{-1}(1/x)$

eg 27 (no calculator) evaluate the expression(s) exactly:
a. $sin^{-1}(-1/2) = -\pi/6$
b. $sec^{-1}(-2) = 2\pi/3$
c. $cot^{-1}(-\sqrt{3}) = \pi/2 + \pi/3 = 5\pi/6$
d. $sin\left(cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) =$
e. Find an algebraic function in terms of x of $y = sin\left(cos^{-1}x\right)$.
f. Use a triangle to show that $arcsin(x) + arccos(x) = \pi/2$.

Derivatives of Inverse Trigonometric Functions

1) $\frac{d}{dx}\left(sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}$
Proof:
Consider $sin\left(sin^{-1}x\right) = x$ for $-1 \leq x \leq 1$. If we differentiate both sides,
$cos\left(sin^{-1}x\right) * \frac{d}{dx}\left(sin^{-1}x\right) = 1$ or $\frac{d}{dx}\left(sin^{-1}x\right) = \frac{1}{cos(sin^{-1}x)} = \frac{1}{\sqrt{1-x^2}}$
by making a triangle with angle θ.
Another Method: $y = sin^{-1}x$ or $sin(y) = x$. If we differentiate both sides, $cos(y)\frac{dy}{dx} = 1$ or
$\frac{dy}{dx} = \frac{1}{cos(y)} = \frac{1}{\sqrt{1-x^2}}$ by making a triangle with angle y.
In general if $u = u(x)$, $\frac{d}{dx}\left(sin^{-1}u(x)\right) = \frac{1}{\sqrt{1-u(x)^2}}\frac{du(x)}{dx}$

2) $\frac{d}{dx}\left(cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}}$
Proof:
Since $arcsin(x) + arccos(x) = \pi/2$ or $arccos(x) = \pi/2 - arcsin(x)$;
$\frac{d}{dx}\left(cos^{-1}x\right) = \frac{d}{dx}\left(\pi/2 - sin^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}}$
In general if $u = u(x)$, $\frac{d}{dx}\left(cos^{-1}u(x)\right) = \frac{-1}{\sqrt{1-u(x)^2}}\frac{du(x)}{dx}$

3) $\frac{d}{dx}\left(tan^{-1}x\right) = \frac{1}{x^2+1}$
Proof:
Let $y = tan^{-1}x$, so $tan(y) = x$. If we take the derivative to both sides, $sec^2(y)y' = 1$, so $y' = cos^2(y) = \frac{1}{x^2+1}$ if we built a triangle with angle y.
In general if $u = u(x)$, $\frac{d}{dx}\left(tan^{-1}u(x)\right) = \frac{1}{u(x)^2+1}\frac{du(x)}{dx}$

4) $\frac{d}{dx}\left(cot^{-1}x\right) = -\frac{1}{x^2+1}$
Proof:
Since $cot^{-1}(x) = \pi/2 - tan^{-1}(x)$, if we differentiate the result follows.

$\frac{d}{dx}\left(sec^{-1}x\right) = \frac{1}{x\sqrt{x^2-1}}$ and $\frac{d}{dx}\left(csc^{-1}x\right) = -\frac{1}{x\sqrt{x^2-1}}$ can be found by differentiating $cos^{-1}\left(\frac{1}{x}\right)$and $sin^{-1}\left(\frac{1}{x}\right)$respectively.

eg 28 A woman is waking at a constant rate of $3ft/s$ towards one post of a $4ft$ wide gate. If her path is perpendicular to one post of the gate and θ is the angle between her path and the line of sight with the other post of the gate, at what rate is the angle increasing when she is $4ft$ from the gate.
Solution: $\theta = cot\frac{s}{4} \rightarrow \frac{d\theta}{dt} = \frac{d\theta}{ds}\frac{ds}{dt} = \frac{-1}{\left(\frac{s}{4}\right)^2+1}\frac{1}{4}(-3) = 3/8rad/s$**.**

Integrals of Trigonometric and Inverse Trigonometric Functions

eg 29 $\int tan(x)dx$; $\int sec(x)dx$; $\int csc(x)dx$

eg 30 $\int \frac{1}{\sqrt{a^2-x^2}}dx = sin^{-1}\left(\frac{x}{a}\right) + c$; $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right) + c$

eg 31 Find the average value of $y = \frac{25}{\sqrt{100-x^2}}$ in $[-5, 5]$ $Ans:\ \frac{5\pi}{6}$

eg 32 $\int \frac{1}{9+x^2}dx$; $\int \frac{\left(tan^{-1}(x)\right)^3}{1+x^2}dx$; $\int \frac{1}{9+4x^2}dx$

eg 33 Find k such that $\int_0^k \frac{1}{4+x^2}dx = \frac{\pi}{8}$ $Ans: 2$

eg 34 Find the volume generated when the area bounded by $y = \frac{1}{\sqrt{x}}, y = 0, x = 1$ and $x = 2$ is rotated about the x-axis. Ans: $\pi\, ln(2)$

7.6 Hyperbolic Functions

Functions obtained from the unit hyperbola $x^2 - y^2 = 1$. They are a combination of exponential functions with properties of trigonometric functions.

They are defined $sinh(x) = \frac{e^x-e^{-x}}{2}$, $cosh(x) = \frac{e^x+e^{-x}}{2}$ where $tanh(x) = \frac{sinh(x)}{cosh(x)} = \frac{e^x-e^{-x}}{e^x+e^{-x}}$
The symmetry properties are:
$sinh(-x) = -sinh(x)$ $cosh(-x) = cosh(x)$

The reciprocal relations are:

$coth(x) = \frac{1}{tanh(x)}$ $sech(x) = \frac{1}{cosh(x)}$ $csch(x) = \frac{1}{sinh(x)}$

From the definition, can be shown that $cosh^2(x) - sinh^2(x) = 1$.

From this we can derive, by dividing it by $cosh^2(x)$ and $sinh^2(x)$ respectively that
$sech^2(x) + tanh^2(x) = 1$ and $-csch^2(x) + coth^2(x) = 1$.

From the definition, we can derive
$cosh(x+y) = cosh(x)cosh(y) + sinh(x)sinh(y)$ or
$sinh(x+y) = sinh(x)cosh(y) + sinh(y)cosh(x)$.

From the above (or by using the definition) it can a derive $sinh(2x) = 2sinh(x)cosh(x)$.

$cosh(2x) = cosh^2(x) + sinh^2(x)$, or $cosh^2(x) = \frac{cosh(2x)+1}{2}$; $sinh^2(x) = \frac{cosh(2x)-1}{2}$

From the definition, and the above identities we can find :
$\frac{d}{dx}sinh(x) = cosh(x)$; $\frac{d}{dx}cosh(x) = sinh(x)$; $\frac{d}{dx}tanh(x) = sech^2(x)$
$\frac{d}{dx}coth(x) = -csch^2(x)$; $\frac{d}{dx}sech(x) = -sech(x)tanh(x)$;
$\frac{d}{dx}csch(x) = -csch(x)coth(x)$

eg 35 If $sinh(x) = 4/3$, Find the value of all the other hyperbolic functions.
Show $(cosh(x) + sinh(x))^k = cosh(kx) + sinh(kx)$ by using the definitions.
Solve $sinh(ln(x)) = 0$; $cosh(x^2 - 1) = 1$
eg 36 Find $\frac{d}{dx}x\,tanh(x)$ $D_x\,sech(x^2)$

$\int_0^1 cosh^2(x)dx$ $\int coth(x)dx$ $\int \frac{sinh(ln(x))}{x}dx$

eg 37 Find the volume of the solid generated by rotating about the x-axis the first quadrant the area under $y = sinh(x)$ $y = 0$ and $x = ln(2)$.
eg. Find the volume of the solid generated by rotating about the y-axis the first quadrant the area under $y = sinh(x)$ and $y = 1$.

eg 38 The shape of a high-voltage line strung between 2 towers (catenary) is given by $y = 20 + \frac{1}{.04}(-1 + cosh(.04x))$ meters. Find how low the line will sag. If the towers are $30m$ apart, how tall are the towers.

7.7 L'Hopital's Rule

eg 39 Consider $\lim_{x\to 0}\frac{x^2-2x}{x^3+x} = \frac{\lim_{x\to 0} x^2-2x}{\lim_{x\to 0} x^3+x} = 0/0$. This limit is called indeterminate of the form $0/0$.
So $\lim_{x\to 0}\frac{x^2-2x}{x^3+x} = \lim_{x\to 0}\frac{x-2}{x^2+1} = -2$ after factoring x.

eg 40 Consider $\lim_{x\to\infty}\frac{x^2-2x}{x^3+x} = \frac{\lim_{x\to\infty} x^2-2x}{\lim_{x\to\infty} x^3+x} = \infty/\infty$. This limit is called indeterminate of the form .
So $\lim_{x\to\infty}\frac{x^2-2x}{x^3+x} = \lim_{x\to\infty}\frac{\frac{1}{x}-\frac{2}{x^2}}{1+\frac{1}{x}} = 0$ after factoring x^3 (divide num. and den.by x^3)

How do we find limits if the functions are not algebraic?

eg 41 Consider $\lim\limits_{x\to 1}\frac{sin(2\pi x)}{x^2-1}$. This limit is of the form $0/0$. If we use the tangent-line approximation of the curves $y = sin(2\pi x)$ and $y = x^2 - 1$ near $x = 1$, we obtain $\begin{cases} sin(2\pi x) & \approx 2\pi(x-1) \\ x^2-1 & \approx 2(x-1) \end{cases}$. This suggest that $\frac{sin(2\pi x)}{x^2-1} \approx \frac{2\pi(x-1)}{2(x-1)} = \pi$ near $x = 1$, so that $\lim\limits_{x\to 1}\frac{sin(2\pi x)}{x^2-1} = \pi$. See graph.

The previous methods will not work if the denominator is zero.

eg 42 Consider $\lim\limits_{x\to 1}\frac{x^2-2x+1}{ln(x)-x+1}$. This limit is of the form $0/0$. Since the derivative of the denominator is zero near $x = 1$, the tangent line approximation fails.

L'Hopital's Rule

Suppose $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}$ is of the form $0/0$ or ∞/∞, $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)} = \lim\limits_{x\to x_0}\frac{f'(x)}{g'(x)} = L$ were L can be zero or infinity and x_0 can be replaced by x_0^+ or x_0^-. Note: L'Hopital's Rule can be applied as many times as needed.

Indeterminate Quotients $0/0$, ∞/∞

eg 43 $\lim\limits_{x\to 1}\frac{sin(2\pi x)}{x^2-1} = \pi$; $\lim\limits_{x\to 1}\frac{ln(x)}{x-1} = 1$; $\lim\limits_{x\to 1}\frac{x^2-2x+1}{ln(x)-x+1}$.

eg 44 $\lim\limits_{x\to 0}\frac{tan(x)-x}{x^3} = \frac{1}{3}$; $\lim\limits_{x\to 1}\frac{x^2-2x+1}{ln(x)-x+1} = -2$

eg 45 $\lim\limits_{x\to\infty}\frac{e^x}{x} = \infty$; $\lim\limits_{x\to 0}\frac{e^h-1}{h} = 0$; $\lim\limits_{x\to 0}\frac{sin(x)}{x} = 1$

Other Quotients

eg 46 $\lim\limits_{x\to 0}\frac{sin(3x)}{1+sin(4x)} = 0$; $\lim\limits_{x\to -\pi}\frac{sin(x)}{1-cos(x)} = 0$ $\lim\limits_{x\to\infty}\frac{sin(x)}{x} = 0$ NL

Indeterminate Products $0\cdot\infty$

eg 47 $\lim\limits_{x\to 0^+}\sqrt{x}\, ln(x) = 0$; $\lim\limits_{x\to\infty} x\, e^{-x} = 0$

Indeterminate Powers $0^0; 1^\infty, \infty^0$

eg 48 $\lim\limits_{x\to 0^+} x^x = 1$; $\lim\limits_{x\to 0^+}(1 + sin(x))^{cot(x)} = e$

eg 49 $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$; $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = 1$;

Indeterminate Differences $\infty - \infty$

eg 50 $\lim\limits_{x\to\infty}\frac{x^2+2}{1+x} - x = -1$; $\lim\limits_{x\to\frac{\pi}{2}^-}(sec(x) - tan(x)) = 0.$