Basic Integration Procedures

1) Basic Substitutions

$a. \int \frac{x+1}{\sqrt{x^2+2x+3}}dx = \sqrt{x^2+2x+3}+c$, by the substitution $u = x^2+2x+3$.

$b. \int \frac{x^2-2x}{x^3-3x^2+1}dx = \frac{1}{3}ln|x^3-3x^2+1|+c$, by the substitution $u = x^3-3x^2+1$.

$c. \int \frac{2}{1+(2x-3)^2}dx = tan^{-1}(2x-3)+c$

Try $\int \frac{1}{e^x+e^{-x}}dx$ ANS: $tan^{-1}(e^x)+c$

2) Completing the Square

$a. \int \frac{dx}{\sqrt{8x-x^2}} = \int \frac{dx}{\sqrt{16-16+8x-x^2}} = \int \frac{dx}{\sqrt{16-(x-4)^2}} = sin^{-1}\left(\frac{x-4}{4}\right)+c$

$b. \int \frac{2}{x^2-6x+13}dx = \int \frac{2}{(x-3)^2+2^2}dx = tan^{-1}\left(\frac{x-3}{2}\right)+c$

$c. \int \frac{dt}{\sqrt{2t-t^2}} = \int \frac{dt}{\sqrt{1-(t-1)^2}} = sin^{-1}(t-1)+c$

Try $\int \frac{dt}{\sqrt{-t^2+4t-3}}$ ANS: $sin^{-1}(t-2)+c$

3) Trigonometric Identities

$a. \int tan(x)csc(x)dx = \int sec(x)dx = ln|sec(x)+tan(x)|+c$

$b. \int csc^2(x)sin(2x)dx = \int 2cot(x)dx = ln|sin(x)|^2+c$

$c. \int (sec(x)+tan(x))^2dx = \int \left(sec^2(x)+2sec(x)tan(x)+tan^2(x)\right)dx =$

$\int \left(sec^2(x)+2sec(x)tan(x)+[sec^2(x)-1]\right)dx = 2tan(x)+2sec(x)-x+c$

Try $\int (sec(x)+cot(x))^2dx$ ANS: $tan(x) = 2ln|csc(x)-cot(x)|-cot(x)-x+c$

4) Reducing Improper Fractions

$a. \int \frac{3x^2-7x}{3x+2}dx = \int \left(x-3+\frac{6}{3x+2}\right)dx = \frac{x^2}{2}-3x+2ln|3x+2|+c$

$b. \int \frac{x^2}{x^2+1}dx = x-tan^{-1}(x)+c$

$c. \int \frac{2x^3}{x^2-1}dx = \int 2x+\frac{2x}{x^2-1}dx = x^2+ln|x^2-1|+c$

Try $\int \frac{4x^3-x^2+16x}{x^2+4}dx$ ANS: $2x^2-x+2tan^{-1}\left(\frac{x}{2}\right)+c$

5) Separating Fractions

$a. \int \frac{1-x}{\sqrt{1-x^2}}dx = \int \left(\frac{1}{\sqrt{1-x^2}}-\frac{x}{\sqrt{1-x^2}}\right)dx = sin^{-1}(x)+\sqrt{1-x^2}+c$

$b. \int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}dx = \int \left(\frac{1}{2\sqrt{x-1}}+\frac{1}{x}\right)dx = \sqrt{x-1}+ln(x)+c$

$c. \int_0^{\pi/4} \frac{1+sin(x)}{cos^2(x)}dx = tan(x)+sec(x)\Big|_0^{\pi/4} = \sqrt{2}$

Try $\int_0^{1/2} \frac{2-8x}{1+4x^2}dx$ ANS: $\frac{\pi}{4}-ln(2)$

6) Substitutions

An integral containing a term $(ax+b)^{\frac{p}{q}}$, let $u=(ax+b)^{\frac{1}{q}}$

$a.\int \frac{dx}{x\sqrt{1+x}} \rightarrow$ let $u=(1+x)^{1/2}$, $u^2=1+x,\ 2udu=dx$ so

$\int \frac{dx}{x\sqrt{1+x}}=\int \frac{2}{u^2-1}du=\ ln|\frac{u-1}{u+1}|+c=ln|\frac{\sqrt{1+x}\ -1}{\sqrt{1+x}\ +1}|+c$

$b.\int \frac{dx}{x^{1/2}-x^{1/4}} \rightarrow$ let $u=x^{1/4}$, $u^4=x,\ 4u^3du=dx;$ so $\int \frac{dx}{x^{1/2}-x^{1/4}}=4\int \frac{u^2}{u-1}du$

$=4\int u+1\frac{1}{u-1}du=4\left(\frac{u^2}{2}+u+ln|u-1|\right)+c=2x^{1/2}+4x^{1/4}+4ln|x^{1/4}-1|+c$

$c.\int \frac{1}{x(x+1)^{\frac{3}{2}}}dx\ =$ let $u=(x+1)^{1/2}$, $u^2=x+1,\ 2udu=dx$ so $\int \frac{1}{x(x+1)^{\frac{3}{2}}}dx$

$=\int \frac{2}{(u^2-1)u^2}du=\int \frac{1}{u-1}-\frac{1}{u+1}-\frac{2}{u^2}du=\ ln|\frac{u-1}{u+1}|+\frac{2}{u}+c=ln|\frac{\sqrt{x+1}\ -1}{\sqrt{x+1}\ +1}|+\frac{2}{\sqrt{x+1}}+c$

Try $\int \frac{1}{\sqrt{x}(x-1)}dx$ ANS: $ln|\frac{\sqrt{x}\ -1}{\sqrt{x}\ +1}|+c$