MAC 2311 Hybrid Calculus I

Sections **4.3,4.5**

**Increasing/decreasing functions**

A continuous function $f $is increasing (inc) on the interval I if $f^{'}\left(x\right)>0$ on I.

A continuous function $f$is decreasing (dec) on the interval I if $f^{'}\left(x\right)<0$ on I.

Eg8. Find the intervals where $f\left(x\right)=x^{3}/3-3x^{2}/2+2x+1$ is increasing/ decreasing

 Ans: from sign chart, inc (-∞,1) U(2,∞); dec (1,2)

**First Derivative Test (FDT) for Relative Extrema**

Suppose $f$is differentiable at every point on an Interval I that contains the critical value c (except perhaps at c itself).

If $f'$ changes sign from positive to negative as x increases through the value c, then $f$ is a local maximum at c.

If $f'$ changes sign from negative to positive as x increases through the value c, then $f$ is a local minimum at c.

If $f'$ does not change sign through c, then $f$ has no local extrema at c.

Eg9. Use the FDT to find the relative extrema of $f\left(x\right)=x^{5}/5-3x^{4}/4+2x^{3}/3+1$ Ans: (0,1) neither max or min; (1,67/60) max; (2,11/15) min

**Second Derivative Test (SDT) for Relative Extrema**

Suppose $ f^{''}\left(x\right)$ is continuous on the open interval I containing c with $ f^{'}\left(c\right)=0$ (at CV)

If $ f^{''}\left(c\right)>0$ then $f$ has a local minimum at c.

If $ f^{''}\left(c\right)<0$ then $f$ has a local maximum at c.

If $ f^{''}\left(c\right)=0$ then the test fails. If the test fails, the first derivative test must be used to find local extrema.

Eg10. Use the SDT to find the relative extrema of $f\left(x\right)=x^{5}/5-3x^{4}/4+2x^{3}/3+1$

Ans: CV at *x*=0,1,2, so$ f^{'}'\left(0\right)=0$, test fails; $f''(1)=-1$, CD or rel max; $f''(2)=4$, CU or rel min. Since SDT fails at x=0, the point (0,1) is neither relative max nor min by the FDT( see Eg9)

Notes: If (c, *f*(c)) is a relative extremum of$ f$, then$ f^{'}\left(c\right)=0$.

The converse is not true. If$ f^{'}\left(c\right)=0$, you may or may not may or may not have a relative extremum.

 **Concavity**

From the intuitive stand point, an arc of a curve is said to be concave up if it has the shape of a cup and concave down if it has the shape of a cap.

If $f'^{'}\left(x\right)>0$ on an open interval I, then the graph of $f$ is concave up (CU) on I.

If $f'^{'}\left(x\right)<0$ on an open interval I, then the graph of $f$ is concave down (CD) on I.

**Test for Concavity (Inflection Points (IP))**

Suppose $ f^{''}\left(x\right)$ exists on the interval I.

If $ f^{''}\left(x\right)>0$ on I, then $f$ is concave up on I.

If $ f^{''}\left(x\right)<0$ on I, then $f$ is concave down on I.

If (c, *f*(c)) is a point on I at which$ f^{''}\left(x\right)=0$, and $ f^{''}\left(x\right)$ changes at c (changes concavity), then (c, *f*(c)) is an inflection point of $f.$

Notes: If (c, *f*(c)) is an inflection point of$ f$, then$ f^{''}\left(c\right)=0$.

The converse is not true. If$ f^{''}\left(c\right)=0$, you may or may not may or may not have an inflection point.

Eg11. Find the inflection points of $f\left(x\right)=x^{4}$ Ans:$ f^{''}\left(0\right)=0$, but there is no IP

Eg12. Find the inflection points of $g\left(x\right)=x^{4}/4-x^{3}$ Ans: (0,0), (2,-4)

**Curve Sketching**

Given; (stands for the function in the numerator and  the denominator)

1. Domain of the function: Possible values of *x.*
2. Intercepts of the function: *xint* solve *N(x)* = 0; *yint= f(0*) .
3. Symmetry: If the function is even (symmetric about the *y-axis*); if the function is odd (symmetric about the *origin*);
4. Asymptotes of the function:

a) Vertical (VA): solve*.* If *x=a* is a vertical asymptote, evaluate  and to find the direction of the curve at the asymptote.

b) Horizontal (HA): Evaluate. This also tells you the behavior of the graph far to the right and left.

1. Interval of Increase/ Decrease Use the first derivative test to find the intervals where *f* *'(x)* >0 (increasing) and where *f* *'(x)* <0 (decreasing). Test *f '(x)* in regions given by zeros, CP and VA in the domain of the function.
2. Local Maximum and Minimum Values: Find the critical points (CP) by finding points where *f* *'(x)* = 0 or *f '(x)* is undefined. Find the *y* coordinate by evaluating *f(x)* at those points. Remember that not all critical points are relative extrema.
3. Concavity and Points of Inflection: Solve *f* *"(x)* = 0 for *x*, to find the possible inflection points. Inflection points are points where concavity changes. Not all points where *f* *"(x)* = 0 are inflection points.

Find where *f (x)* is concave up (CU) or concave down (CD) by testing *f ''(x)* in regions given by zeros, CP and VA in the domain of the function. If you evaluate at the CP, you obtain  (CU/relative min); (CD/ relative max).

1. Sketch the curve without the use of a graphics calculator. Check with a grapher.

Note: Start your preliminary sketch as you find the information about the graph in parts A-G.

Examples:

1. Sketch the graph *of f(x)* = *x3 -*3*x*

1. Domain: all *x.*
2. *xint*  *x(x2-*3*)* = 0  *x* = 0, *x=*; *yint* = 0.
3. the function is odd (symmetric about the *origin*)
4. No asymptotes. = =
5. Increasing when *y'* = *3x2* -3 > 0 on **, and decreasing when

*y'* = *3x2* -3 < 0 on (-1,1).

1. Critical numbers at *y'* = *3x2* -3 = 0 *(x* = *-*1); *(x* = l*,*). ** is changes from positive to negative at *x* = *-*1 ( I(-1)=2 is a local maximum) and changes from negative to positive at *x* = 1 ( f(1)=-2 is a local minimum)
2. = 0 at (0,0). > 0 for *x>*0 (CU); < 0 for *x*<0 (CD). Since concavity changes, (0,0) is an IP. |1> 0 (CU/relative min.); |-1< 0 (CD/relative max.)
3. 

2. Sketch the graph of 

1. Domain 
2. *xint*  *x* = 0; *yint* = 0.
3.  the function is even (symmetric about the *y-axis*)
4. VA, , ==; ==HA, =1 =1
5. Increasing when  >0 on, and decreasing when

 <0 on,

1.  = 0 *(x* = 0*,y* = 0); undefined (not in the domain of *f(x)*), local maximum at *f*(0)=0, no local minimum.
2. **; no IP. *y''* = > 0 for *x>*0 for *x* <-1 or x >1 (concave up)

 *y"* = <0 for -1<x<1, (concave down).

1. 

3. Sketch the graph of $f\left(x\right)=\frac{1}{x^{2}+1} $

1. Domain: all *x*
2. *x*int none, *y*int =1
3. Symmetry: even
4. VA none; HA $\lim\_{x \to \infty }\left(1/(x^{2}+1)\right)=0$
5. Interval of increase/decrease $f^{'}(x)=-2x/(x^{2}+1)^{2}$, CP *x* =0

$f^{'}\left(x\right)>0$ for *x* < 0; $f^{'}\left(x\right)<0$ for *x* > 0, inc(-∞,0), dec (0,∞)

1. Concavity and Inflection points: $f^{''}(x)=(6x^{2}-2)/(x^{2}+1)^{3}$

CU (-∞,-1/√3) U (1/√3, ∞); CD (-1/√3,1/√3), IP ( ±1/√3, ¾)

1. Graph (label all points)
2. 

4. Sketch the graph of $f\left(x\right)=4x^{3}-x^{4}$

1. Domain: all *x*
2. *x*int =0,4, *y*int =0
3. Symmetry: none
4. VA none; HA none
5. Interval of increase/decrease $f^{'}(x)=12x^{2}-4x^{3}=0$, CP (0,0), (3,27)

$f^{'}\left(x\right)>0$ in (-∞,3); $f^{'}\left(x\right)<0$ in (3,∞), inc(-∞,3), dec (3,∞)

1. Concavity and Inflection points: $f^{''}\left(x\right)=24x-12x^{2}=0, x=0,x=2$

CU (0,2); CD (-∞,0) U (2, ∞); IP ( 0,0), (2, 16)

1. Graph (label all points) Scale: *x*=1, y=3
2. 

**Exercises:**

1. Find the critical points
2. Test for symmetry
3. Find the intervals on which *f* is increasing or decreasing
4. Find the intervals where *f* is concave up or concave down.
5. Find the inflection points if any.
6. Use the information in parts a-e to sketch the graph. Check with a grapher.

1) 

2) 

3) 

4) 

5) 

6) 

7) 

8) 

9) 

10) 

**Answers:**

1. Inc. ; Dec. ; local max ; local min 

 CU (; CD ; IP 

1. Inc. ; Dec. ; local max ;

 local min ; CU ; CD ; IP 

1. Inc. ; Dec. ; local max ;

 local min ; CU ; CD ;

 IP , 

1. Inc. ; Dec. ; local max ; local min 

 CU (; CD ; IP 

1. Inc. ; Dec. ; local max ;

 local min ; CU ; CD ;

 IP 

1. *y*-int :0; *x*-int :0; VA: *x=*1; HA: *y* =1; Dec: ; CU ; CD ; IP: none
2. *y*-int :-1/9; VA: ; HA: ; Inc. ; Dec: ; local max ; CU ; CD ; IP: none
3. *y*-int :1; *x*-int :1; VA: **; HA:; Dec: ; CU ; CD ; IP: none
4. *y*-int :0; *x*-int :0; VA: None; HA: None ; Inc: ; Dec: ; local max ; local min ; CU ; CD ; IP: ;;
5. *y*-int :0; *x*-int :0; Inc: ; Dec: ; local max ; local min ; CU ; CD ; IP: ;;