MHF 2300 – Spring 2007

The Proof Portfolio Project

Description of the Portfolio Project

During the semester, ten problems will be posed to all students. Each student will work on these problems and submit proposed solutions to the professor at the end of the semester. You may bring each of your portfolio problems to the professor one time to be critiqued. The professor will make recommendations about these and give a "temporary grade" for the problem. You then have the opportunity to rewrite and resubmit your problem for the final portfolio.

Important guidelines and rules for the Portfolio Project

- You may not discuss the portfolio problems with anyone except the instructor of the course.
- You may not use any sources to help complete the portfolio problems other than the textbook.
- You may hand in a given portfolio problem to the professor one time to be critiqued.
- No more than two portfolio problems may be submitted for review on a given day.
- The last date to have a portfolio problem critiqued is the beginning of class on Wednesday April 11, 2007.
- The final portfolio is due no later than the beginning of class on Wednesday April 18, 2007.

Grading of the Portfolio Project

The portfolio will be worth a total of 120 points. Each problem will be worth 10 points (for a total of 100 points). In addition, there will be 20 points possible for submission of proofs for review by the professor. To be eligible for the 20 points, a student must do all of the following:

- Submit the first draft of a portfolio problem by Monday January 29, 2007
- Submit the first draft of a second portfolio problem (different than the first) by Monday February 5, 2007
- Submit the first draft of a third portfolio problem (different than the first two) by Monday February 12, 2007
- Submit the first draft of a fourth portfolio problem (different than the first three) by Monday February 19, 2007
- Submit the first draft of a fifth portfolio problem (different than the first four) by Monday February 26, 2007
- Submit the first draft of a sixth portfolio problem (different than the first five) by Monday March 5, 2007
- Submit the first draft of a seventh portfolio problem (different than the first six) by Monday March 19, 2007
- Submit the first draft of a eighth portfolio problem (different than the first seven) by Monday March 26, 2007
- Submit the first draft of a ninth portfolio problem (different than the first eight) by Monday April 2, 2007
- Submit the first draft of a tenth portfolio problem (different than the first nine) by Monday April 9, 2007

If you meet these ten deadlines, your score for this part of the portfolio will be 20 out of 20 points. For each deadline that is missed, 2 points will be deducted from your score. Please note that these deadlines are independent of each other. For example, if you miss the deadline for submitting a second portfolio draft (February 5, 2007), then you still must have three drafts in by February 12, 2007 to meet the third deadline.

Honor System

All work that you submit for the Portfolio Project must be your own work. **This means that you may not discuss the portfolio project with anyone except the instructor of the course and may not use any resources other than the textbook, with the exception that you may look up relevant review material from past mathematics courses.** If you do so, you should cite the book you used and the specific pages you used. You are not, however, permitted to go looking for completed solutions to these problems in any other texts or resources. In particular, using the Internet is completely off limits. Evidence of using Internet sources in your work will result in a minimum penalty of failure of the assignment.

This will also provide me with information regarding how students are doing with each problem. So, if I find that a particular problem is causing more difficulties that anticipated, I can send an email message to all students with hints or points of clarification for that problem.

Following are some (anticipated) questions about this Portfolio. The answers to these questions contain some very important requirements and guidelines for the Portfolio Project.

What other requirements are there for my Portfolio Problems?

The solution for each problem must be written using complete sentences and according to the writing guidelines specified in the text. It must be neat, well organized, and easy to read. Proper grammar, proper sentence and paragraph structure, and correct spelling are necessities.

What happens if I submit an incorrect or incomplete solution?

The professor will return your problem and indicate if it is ready for your Portfolio or if it needs more work. When you submit a solution for a problem before the last day for review, you are asking the professor, "Is this good enough for my Portfolio?" Only the problems turned in by the beginning of class on Wednesday April 18, 2007 will determine your score on those two portions of the portfolio.

When can I submit a proposed solution for a problem?

You may submit a problem for review any time by the beginning of class on Wednesday April 11, 2007**.**

Should I wait and submit all my problems for review on the last day?

NO!! As soon as you have a proposed solution for a problem, you should write your solution and submit it for review. To encourage this, no more than two portfolio problems may be submitted for review on a given day. **Begin working on your Portfolio Project Immediately!!!**

Can I work with someone else or sources other than the textbook?

The only person you can discuss these problems with is the instructor for the course and the only resource you may use is the textbook. ***Plagiarism is not acceptable*** and will not be tolerated. No credit will be given for the solutions of problems in which plagiarism is involved.

What criteria will be used to judge my proofs?
A proof must be logically and mathematically correct. In addition, it must be written according to the course guidelines as developed in the text and discussed in class.

How will my grade for a given problem be determined?
Each problem in your portfolio will be graded on a 10-point scale with the only possible grades being 10, 9, 6, 3, or 0 points. There will be little partial credit because of the opportunity to submit problems for review, to re-write, and to re-submit. In order to receive full credit for a problem, your solution must be correct, complete, and well written with no spelling or grammatical errors. Following is a description of the 10-point scale for grading each problem:

Points	**Description**
10	The proof is correct and written according to the guidelines in the text plus those that follow.
9	The proof is correct but there is a writing mistake.
6	The proof is essentially correct but the solution is not written according to the guidelines.
3	Significant progress has been made in developing and writing a proof for the theorem.
0	Little or no progress has been made in developing a proof for the theorem.

How should I start working on a particular problem?
Before beginning your proof or solution of the problem, you should make a clear statement of exactly what it is that is given in problem (the assumptions) and what is to be proven (the goal). That is, you should analyze the theorem or problem by carefully examining what is given or assumed and precisely what it is that will be proven. In this analysis, you should include any relevant definitions that are needed to clarify the statement of the problem. You should also elaborate on the assumptions made and the strategies that can be used to prove what it is that you are trying to prove. If it is appropriate, you may also include some examples to illustrate the problem.

What are the writing guidelines for writing the solutions of the Portfolio Problems?
To receive full credit, the solution of a portfolio problem must be of collegiate quality and follow the writing guidelines for this course that are given in the textbook. This means that, in addition to demonstrating mastery of the subject matter, the solution should be neat and easy to read, well organized, and use proper grammar and spelling. In addition, a solution must meet the following guidelines:

- You should begin your presentation with a carefully worded statement of the problem. Do not use phrases such as "Show that" or "Prove that". You should state the problem using simple declarative sentences. Following is a typical textbook problem.

Prove that if n is an integer and n^2 is odd, then n is odd.

If you were writing a solution to this problem for one of these writing assignments, you should begin in the following manner:

Theorem: If n is an integer and n^2 is an odd integer, then n is an odd integer.

- All calculations and algebraic manipulations must be clearly shown. By doing so, both you and your professor can follow the process you used to obtain an answer. Without a step-by-step presentation, it may be impossible to understand your solution, or if a mistake is made, it may be impossible to determine where a mistake was made.

- You might start your solution with a short discussion of the strategy that you will use. This is required if you use an indirect method of proof such as a proof by contradiction or use of the contrapositive of a statement. In addition, you should conclude any proof with a statement of what has been proven, or minimally, that the proof is now complete.

Problem One

Suppose that a, b, x, y, n are integers. Prove: If n divides a and n divides b then n divides $ax+by$.

Problem Two

Prove that if n is an integer then n^2 has the form $3k$ or $3k+1$.

Problem Three

Prove the lemma: For each integer n, 3 divides n if and only if 3 divides n^2.

Use the lemma to prove that $\sqrt{3}$ is an irrational number.

Problem Four

An integer n is called a **square** provided there exists an integer m such that $n = m^2$. Prove that no square has the form 4k + 3 where k is an integer. In other words prove:

If n is a square then n does not have the form 4k+3.

Problem Five

Prove or disprove each of the following:

- For any sets A, B, and C that are subsets of a universal set U, if $A \cap B \neq \varnothing$ and $B \cap C \neq \varnothing$ and $A \cap C \neq \varnothing$ then $A \cap B \cap C \neq \varnothing$.

- For any sets A, B, and C that are subsets of a universal set U, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Problem Six

Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of real numbers satisfying: $a_1 = 4$ and $a_2 = 5$ and for all integers $n \geq 3$, $a_n = \frac{5a_{n-1}}{6} + \frac{3}{a_{n-2}}$. Prove that for every natural number n, $3 \leq a_n \leq 6$.

Problem Seven

Which natural numbers can be written as a sum of 3's and 5's. For example 9 is such a number since 9 = 3 + 3 + 3 and 13 is one since 13 = 3 + 5 + 5. However 7 is not one since there is no combination of 3 and 5 that give a sum of 7. State and prove a conjecture.

Problem Eight

Recall that Z is the set of integers and Q is the set of rational numbers. Let $A = \{(m, n) \mid (m, n) \in Z \times Z,\ n \neq 0\}$ and let $f : A \to Q$ be defined by $f(m, n) = \frac{m+n}{n}$.

Is f an injection? Justify your answer.

Is f a surjection? Justify your answer.

Problem Nine

Suppose A and B are sets and $f : A \to B$ is a function. If $S \subset B$ then $f^{-1}(S)$ is a subset of A defined by $f^{-1}(S) = \{x \in A \mid f(x) \in S\}$. Using the terminology in chapter 6 we can say that $f^{-1}(S)$ is the set of pre-images of the elements of S.

Example: Recall that R is the set of real numbers. Consider the function $f : R \to R$ defined by $f(x) = x^2$. If $S = \{3\}$ then $f^{-1}(S) = \{\sqrt{3}, -\sqrt{3}\}$. On the other hand, if the set $S = \{y \in R \mid 1 \leq y \leq 4\}$ then $f^{-1}(S) = \{x \in R \mid -2 \leq x \leq -1 \text{ or } 1 \leq x \leq 2\}$.

Prove the following: Suppose A and B are sets and $f : A \to B$ is a function. If S and T are subsets of B then $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$ and $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

Problem Ten

Prove that for any list of three consecutive natural numbers say $n-1$, n, and $n+1$ where $n-1$ and $n+1$ are both prime numbers with $n > 4$, the number n is divisible by 6.

Comment: Any pair of prime numbers that are two units away from each other on the number line are called <u>twin primes</u>. The natural number between a pair of twin primes is called the <u>midpoint</u>. In this problem, $n-1$ and $n+1$ are twin primes with n being the midpoint. With these definitions in mind, another way of stating the problem would be to prove that the midpoint of any pair of twin primes greater than 4 is always divisible by 6.

Alternate Problem: This problem can be used in place of any one of the preceding ten problems. If you decide to do this problem, you need to state which problem you wish to replace by this one.

Definition: Let H and K be subsets of the real numbers. The set H is said to be <u>nowhere dense</u> in the set K if every open interval intersecting K contains an open interval which intersects K but not H.

Remarks: Recall that an open interval (p, q) is the set of all real numbers strictly between p and q.

Also, when we say that the open interval (p, q) contains another open interval (a, b) we mean that $(a, b) \subset (p, q)$. Furthermore, an open interval (p, q) intersecting K means that $(p, q) \cap K \neq \emptyset$.

Theorem: If H is nowhere dense in K, then every subset of H is nowhere dense in K.
