The Well-Ordering Principle Implies The Principle of Mathematical Induction.

Proof: Let $P(n)$ be a predicate whose truth depends on $n \in N$ where N is the set of natural numbers. Let us assume that $P(1)$ is true AND $P(k)$ implies $P(k+1)$ for all $k \geq 1$. By way of contradiction, suppose that there exists $n \in N$ such that $P(n)$ is false. We define the set $S = \{x \in N : P(x) \text{ is false}\}$. It is clear from this definition that $S \subseteq N$. We also remember that $n \in S$ so that $S \neq \varnothing$. Since S is a non-empty subset of N, the well-ordering principle guarantees us that S has a least element. This means that there exists $r \in S$ such that $r \leq x$ for all $x \in S$. We can rest assured that $r \neq 1$ since our hypothesis states that $P(1)$ is true. It follows that $r \geq 2$. We conclude that $r-1 \notin S$, for otherwise, $r \leq r-1$ which is impossible since we already know that $r-1 < r$. By definition of the set S, we conclude that $P(r)$ is false and $P(r-1)$ is true. This, in turn, tells us that $P(r-1)$ does not imply $P(r)$ which contradicts the second part of our hypothesis with $k = r-1$. It follows that $P(n)$ is true for all $n \in N$. This establishes the Principle of Mathematical Induction.

The Principle of Mathematical Induction Implies The Well-Ordering Principle.

Lemma: Any non-empty finite subset of the natural numbers has a least element.

Proof: Let N denote the set of natural numbers. For all $n \in N$, let $P(n)$ be the predicate: "If $B \subseteq N$ with $|B| = n$ then there exists $q \in B$ such that $q \leq x$ for all $x \in B$." We proceed to prove the lemma by induction on n.

Base Step: Let $B \subseteq N$ with $|B| = 1$. Since $B \neq \varnothing$ we can pick $q \in B$ and see with certainty that $q \leq q$, making q the least element of B. This completes the base step.

Induction Step: Let $k \geq 1$ be given and assume that $P(k)$ is true. This is our induction hypothesis. We need to show that $P(k+1)$ is true. To this end, let $B \subseteq N$ with $|B| = k+1$. We need to show that B has a least element. Since $B \neq \varnothing$ we may pick $q \in B$ and consider the set $B - \{q\}$. We know that $|B - \{q\}| = k$ since we are "subtracting" only one element from B. We also know that $B - \{q\} \subseteq N$ and so we apply the induction hypothesis and obtain $p \in B - \{q\}$ such that $p \leq x$ for all $x \in B - \{q\}$. If $p \leq q$ then $p \leq x$ for all $x \in B$ and this makes p the least element of B. On the other hand, if $q < p$ then $q < p \leq x$ for all $x \in B - \{q\}$. But certainly we know that $q \leq q$, so it follows that $q \leq x$ for all $x \in B$ and this makes q the least element of B. In either case, we have shown that B has a least element. This completes the induction step.

By the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in N$. This means that every non-empty finite subset of natural numbers has a least element.