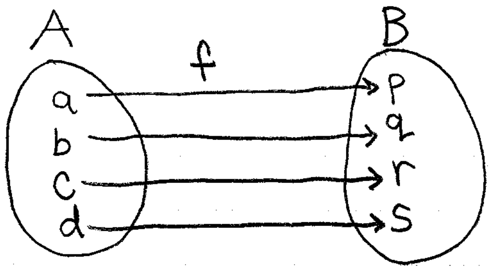
Exam 2 KEY

① a) $A \cup (B \cap C) = \{1,2,4,5,6\}$

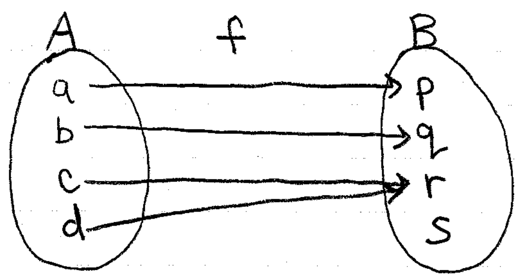
b) $A - B = \{1,4\}$

c) $(A \cap C)^c = \{1,3,4,5,6,7,8,9,10\}$

② a)



b)



③ a) $[4] \oplus [5] = [1]$

b) $[3] \otimes [5] = [7]$

c) $[3]^3 = [3]$

d) $[a] = [4]$ and $[b] = [2]$

④ a) False b/c 13 is not an ordered pair.

b) False b/c π is not an ordered pair.

c) True b/c $10 \in \mathbb{Z}$ and $\pi \in \mathbb{R}$

d) False b/c $\pi \notin \mathbb{Z}$

⑤ a) and

b) and

c) or

⑥ a) $d(15)=4$ since $1,3,5,15$ are the only divisors

b) Yes since $2 \cdot 2 = 4$ and $3,5$ are prime numbers.

c) No b/c $d(8)=4 \neq 2 \cdot 3 = 6 = d(2) \cdot d(4)$

d) No b/c $d(15)=d(8)=4$ and $15 \neq 8$.

⑦ Let $p \in (A \cup B)^c$. Then $p \notin (A \cup B)$ implies $p \notin A$ and $p \notin B$ by 5b on test. This implies $p \in A^c$ and $p \in B^c$ implies $p \in A^c \cap B^c$ implies $(A \cup B)^c \subseteq A^c \cap B^c$

To show the other inclusion, let $p \in A^c \cap B^c$. Then $p \in A^c$ and $p \in B^c$ implies $p \notin A$ and $p \notin B$ implies $p \notin (A \cup B)$ by 5b on test. This implies $p \in (A \cup B)^c$ implies $A^c \cap B^c \subseteq (A \cup B)^c$

Both parts above together imply that $(A \cup B)^c = A^c \cap B^c$.

(8) Let $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = 3x+5$. To show that f is a bijection we need to show that f is surjective and injective. To show f is injective let $p, q \in \mathbb{R}$ and assume $f(p) = f(q)$. This means that $3p+5 = 3q+5$ which in turn implies that $3p = 3q$ by subtracting 5 from both sides of the former equation. Now dividing the latter equation by 3 we obtain that $p = q$. We have thus shown that for all $p, q \in \mathbb{R}$ $f(p) = f(q)$ implies $p = q$ and so f is injective. Now to show that f is surjective let $q \in \mathbb{R}$. We need to find $p \in \mathbb{R}$ so that $f(p) = q$. I claim that $p = \frac{q-5}{3}$ will work. First note that $p \in \mathbb{R}$ since real numbers are closed under subtraction and multiplication i.e. $p = \frac{1}{3} \cdot (q-5)$ and both $\frac{1}{3}$ and 5 are real numbers. Now $f(p) = 3p+5 = 3\left(\frac{q-5}{3}\right)+5 = q-5+5 = q$

We have thus shown that for all $q \in \mathbb{R}$ there exists a $p \in \mathbb{R}$ such that $f(p) = q$. This means f is surjective. Now that we have f both injective and surjective it follows that f is bijective. This completes the proof.

(9) Let $P(n)$ be $3+6+9+\cdots+3n = \frac{3n(n+1)}{2}$

We first show $P(1)$ is true. In our case we have $3(1) = \frac{3(1)[(1)+1]}{2} = \frac{3 \cdot 2}{2} = 3$

Next let $k \in \mathbb{N}$ and assume $P(k)$ is true. That is we assume the equation

$3+6+9+\cdots+3k = \frac{3k(k+1)}{2}$ holds. We need to show $P(k+1)$ is true.

Look at $\underbrace{3+6+9+\cdots+3k}+3(k+1)$. Using the inductive hypothesis this sum

equals $\frac{3k(k+1)}{2} + 3(k+1) = \frac{3k(k+1)}{2} + \frac{6(k+1)}{2} = \frac{3k^2+9k+6}{2}$

$= \frac{(3k+3)(k+2)}{2} = \frac{3(k+1)[(k+1)+1]}{2}$. So $P(k+1)$ is true

By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.