**Page 303**
**Problem #13**

Part a) Let $f : Z \times Z \to Z$ be defined by $f(m,n) = 2m + n$. I claim that $f$ is a surjection and $f$ is not an injection.

Proof: Let $p \in Z$. We need to find a pre-image for $p$. That is, we need to find an element $q \in Z \times Z$ such that $f(q) = p$. The element $q$ has the form $(x, y)$ where $x$ and $y$ are integers.

Case 1: Suppose that $p$ is even. Then we have that $p = 2r$ for some integer $r$. Equivalently, $p = 2r + 0$. So by setting $x = r$ and $y = 0$ we have $q = (r, 0)$ and $f(q) = p$ as required.

Case 2: Suppose that $p$ is odd. Then we have that $p = 2s + 1$ for some integer $s$. So by setting $x = s$ and $y = 1$ we have $q = (s,1)$ and $f(q) = p$ as required.

In either case, it follows that $f$ is a surjection.

To prove that $f$ is not an injection, we have to find elements $p$ and $q$ both members of $Z \times Z$ such that $p \neq q$ and $f(p) = f(q)$. To this end let $p = (3,1)$ and $q = (-1,9)$. We can see that $p \neq q$ since $3 \neq -1$ and $1 \neq 9$. Now look at the equation below.

$$f(p) = f[(3,1)] = 2(3) + 1 = 7 = 2(-1) + 9 = f[(-1,9)] = f(q)$$

This equation shows that $f(p) = f(q)$ even though $p \neq q$. This shows that $f$ is not injective. This concludes the proof.

Part b) Let $g : Z \times Z \to Z$ be defined by $g(m,n) = 6m + 3n$. I claim that $g$ is not a surjection and $g$ is not an injection.

Proof: To show that $g$ is not a surjection, we need to find an element $q \in Z$ such that $g(p) \neq q$ for all $p \in Z \times Z$. If we take $q = 5$, then let $p = (x, y) \in Z \times Z$. We need to show that $g(p) \neq 5$. By way of contradiction, suppose for the moment that $f(p) = 5$. This means that $6x + 3y = 5$. This last equation implies that $2x + y = 5/3$. We know that $x$ and $y$ are integers and that integers are closed under addition and multiplication. This means that $2x + y$ must be an integer yet we know that $5/3$ is not an integer. This is a contradiction, hence $g(p) \neq 5$ which is what we were required to show.

To prove that $g$ is not an injection, we have to find elements $p$ and $q$ both members of $Z \times Z$ such that $p \neq q$ and $g(p) = g(q)$. To this end, let $p = (1, 2)$ and $q = (-3,10)$. We can see that $p \neq q$ since $1 \neq -3$ and $2 \neq 10$. Now look at the equation below.

$$g(p) = g[(1,2)] = 6(1) + 3(2) = 12 = 6(-3) + 3(10) = g[(-3,10)] = g(q)$$

This equation shows that $g(p) = g(q)$ even though $p \neq q$. This shows that $g$ is not injective. This concludes the proof.