Lemma: For all $n \geq 0$, $10^n \equiv 1 \pmod 9$

Proof: By induction (exercise)

Thm: If $n \in \mathbb{N}$ and $S(n)$ denotes the sum of digits of n, then $9 | n$ if and only if $9 | S(n)$.

Proof: Let $n \in \mathbb{N}$. We don't know how many digits n has since n is arbitrary. We do know that the number of digits must be finite. Say that n has k digits, then $n = a_k a_{k-1} \cdots a_3 a_2 a_1 a_0$ where a_0 is the ones place, a_1 is the tens place and a_2 is the hundreds place ect... This means n can be written in the form:

$$n = (a_k \times 10^k) + (a_{k-1} \times 10^{k-1}) + \dots + (a_2 \times 10^2) + (a_1 \times 10) + (a_0 \times 10^0)$$

Since both the left and right-hand sides of the above equation are equal, they both belong to the same equivalence class modulo 9. Thus

$$\begin{aligned}
[n] &= [(a_k \times 10^k) + \dots + (a_0 \times 10^0)] \\
&= [(a_k \times 10^k)] \oplus \cdots \oplus [(a_0 \times 10^0)] \\
&= ([a_k] \otimes [10^k]) \oplus \cdots \oplus ([a_0] \otimes [10^0]) \quad \Big\} \text{ by definition on pg. 387} \\
&= ([a_k] \otimes [1]) \oplus \cdots \oplus ([a_0] \otimes [1]) \quad \Big\} \text{ by the results of the lemma} \\
&= ([a_k \cdot 1]) \oplus \cdots \oplus ([a_0 \cdot 1]) \\
&= [a_k] \oplus \cdots \oplus [a_0] \\
&= [a_k + \dots + a_0] \\
&= [S(n)]
\end{aligned}$$

We showed that $[n] = [S(n)]$ so that $n \equiv S(n) \pmod 9$ since we are dealing with equivalence classes modulo 9. This mean $9 | (n - S(n))$ or there exists $k \in \mathbb{Z}$ such that $n - S(n) = 9k$.

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$(\Rightarrow)$ First if we assume that $9 \mid n$ then there exits an integer p such that

$$n = 9p.$$

We need to show that $9 \mid S(n)$. This means we need to find an integer q such that $S(n) = 9q$. To this end, since $n - S(n) = 9k$ then

$$S(n) = n - 9k$$
$$S(n) = 9p - 9k \quad \text{since } n = 9p.$$
$$S(n) = 9(p-k)$$

so $q = p - k \in \mathbb{Z}$ since integers are closed under subtraction.
This proves that $9 \mid S(n)$ whenever $9 \mid n$.

$(\Leftarrow)$ On the other hand, assume that $9 \mid S(n)$ then there exists an integer p such that

$$S(n) = 9p$$

We need to show that $9 \mid n$. This means we need to find an integer q such that $n = 9q$. To this end, since $n - S(n) = 9k$ then

$$n - 9p = 9k$$
$$n = 9p + 9k$$
$$n = 9(p+k)$$

so $q = p + k \in \mathbb{Z}$ since integers are closed under subtraction.
This proves that $9 \mid n$ whenever $9 \mid S(n)$.

We have thus shown that $9 \mid n$ if and only if $9 \mid S(n)$.

ex Let $n = 3573$ then $S(n) = 3+5+7+3 = 18$.

Since $9 \mid 18$ then by what we just proved we conclude that $9 \mid 3573$.

Note: In our proof, $[a] = [b]$ if and only if $a \equiv b \bmod 9$