MAC 2312 Trig Substitution

Principle II: If $y = \frac{a}{b}\sin u$ then $\sqrt{a^2 - b^2 y^2} = a\cos u$

Pg. 508 #2

$\int x^3\sqrt{9-x^2}\,dx$ Since $\sqrt{9-x^2} = \sqrt{3^2 - 1^2 \cdot x^2}$ we have $a=3;\ b=1;\ y=x$

Let $\underset{y}{x} = \frac{\overset{a}{3}}{\underset{b}{1}}\sin u$ then $\sqrt{9-x^2} = 3\cos u$ and $x^3 = 3^3\sin^3 u$

$dx = 3\cos u\,du$

$$= \int \underbrace{(3^3\sin^3 u)}_{x^3}\underbrace{(3\cos u)}_{\sqrt{9-x^2}}\underbrace{(3\cos u\,du)}_{dx}$$

save out.

$$= 3^5\int \sin^3 u\cos^2 u\,du = 3^5\int \sin^2 u\cos^2 u \cdot \sin u\,du = 3^5\int (1-\cos^2 u)\cos^2 u \cdot \sin u\,du$$

$$= 3^5\int (\cos^2 u - \cos^4 u)\cdot \sin u\,du = 3^5\int (w^2 - w^4)(-dw) = 3^5\int w^4 - w^2\,dw$$

Let $w = \cos u$

$-dw = \sin u\,du$

$$= 3^5\left[\frac{1}{5}w^5 - \frac{1}{3}w^3\right] + C$$

$$= 3^5\left[\frac{1}{5}\cos^5 u - \frac{1}{3}\cos^3 u\right] + C$$

$$= 3^5\left[\frac{1}{5}\left(\frac{\sqrt{9-x^2}}{3}\right)^5 - \frac{1}{3}\left(\frac{\sqrt{9-x^2}}{3}\right)^3\right] + C$$

$$= \frac{1}{5}(9-x^2)^{5/2} - 3(9-x^2)^{3/2} + C$$

Principle 2: If $y = \frac{a}{b}\tan u$ then $\sqrt{a^2+b^2y^2} = a\sec u$

completing the square

Pg. 508 # 24 $\displaystyle\int \frac{dx}{\sqrt{x^2-6x+13}} = \int \frac{dx}{\sqrt{(x-3)^2+2^2}} = \int \frac{dx}{\sqrt{2^2+1^2\cdot(x-3)^2}}$ so $a=2$; $b=1$ and $y = x-3$

Let $x-3 = \frac{2}{1}\tan u$ then $\sqrt{(x-3)^2+4} = 2\sec u = \sqrt{x^2-6x+13}$

$x = 2\tan u + 3 \longrightarrow \tan u = \frac{x-3}{2}$

$dx = 2\sec^2 u\, du$

memorized fact

$$= \int \frac{2\sec^2 u\,du}{2\sec u} = \int \sec u\,du = \ln\left|\sec u + \tan u\right| + c$$

$$= \ln\left|\frac{\sqrt{x^2-6x+13}}{2} + \frac{x-3}{2}\right| + c$$

logarithm rule

$$= \ln\left|\sqrt{x^2-6x+13} + x - 3\right| - \ln(2) + c$$

$$= \ln\left|\sqrt{x^2-6x+13} + x - 3\right| + K$$

where $K = c - \ln(2)$

Principle 3: If $y = \frac{a}{b}\sec u$ then $\sqrt{b^2y^2 - a^2} = a\tan u$

example $\displaystyle\int \frac{\sqrt{25x^2 - 81}}{x}\,dx$ Since $\sqrt{25x^2-81} = \sqrt{5^2x^2 - 9^2}$ then $b=5$; $a=9$; $y=x$

Let $x = \frac{9}{5}\sec u$ then $\sqrt{25x^2-81} = 9\tan u$

$u = \sec^{-1}\left(\frac{5x}{9}\right)$

$dx = \frac{9}{5}\sec u \tan u \, du$

$$= \int \frac{9\tan u}{\frac{9}{5}\sec u}\,\frac{9}{5}\sec u \tan u\,du = 9\int \tan^2 u\,du$$

$$= 9\int(\sec^2 u - 1)\,du$$

$$= 9\left[\tan u - u\right] + C$$

$$= 9\left[\frac{\sqrt{25x^2-81}}{9} - \sec^{-1}\left(\frac{5x}{9}\right)\right] + C$$

$$= \sqrt{25x^2-81} - 9\sec^{-1}\left(\frac{5x}{9}\right) + C$$ for $x > 0$