Quiz 14

If $a_n = \dfrac{\ln(n)}{n^2}$ then $f(x) = \dfrac{\ln(x)}{x^2}$ and $f'(x) = \dfrac{1 - 2\ln x}{x^3}$ (f is pos. and cont. for $x > 1$)

The only critical point of f is $x = \sqrt{e}$. The sign chart $\begin{array}{c|c|c} & + & - \\ 0 & & \sqrt{e} \end{array}$

shows that f decreases on $(\sqrt{e}, \infty)$. Now we use the integral test to

show $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ converges. To this end, look at $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^2}\,dx$.

Using integration by parts with $f = \ln x$, $g = \frac{-1}{x}$, $f' = \frac{1}{x}$, $g' = \frac{1}{x^2}$ we obtain:

$$\lim_{b\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left[\left(-\frac{\ln b}{b} - \frac{1}{b}\right) - (0 - 1)\right]$$

$$= -\lim_{b\to\infty}\left(\frac{\ln b}{b} - \frac{1}{b}\right) + 1$$

L'Hospital's Rule $$= -\left(\lim_{b\to\infty} \frac{\frac{1}{b}}{1} - \lim_{b\to\infty} \frac{1}{b}\right) + 1$$

$$= -(0 - 0) + 1$$

$$= \underline{1} < \infty$$

By the integral test, $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ converges since $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$ converges.