MAC 2311 Calculus with Analytic Geometry I

TEST 3

Name KEY

Score 72

1. Use the closed interval method to find **both** the absolute minimum and absolute maximum values of the function $f(x) = \dfrac{\cos(x)}{2+\sin(x)}$ on the interval $[0, 2\pi]$. Express your final answers in exact form which means **do not** use decimals. (12 points)

① $f'(x) = \dfrac{(2+\sin x)(-\sin x) - (\cos x)(\cos x)}{(2+\sin x)^2} = \dfrac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2}$

$= \dfrac{-2\sin x - 1}{(2+\sin x)^2}$

Since $(2+\sin x)^2$ never $= 0$ the critical values must occur when $-2\sin x - 1 = 0$. This happens if and only if $\sin x = \frac{-1}{2}$. Hence $x = \frac{7\pi}{6}$ and $x = \frac{11\pi}{6}$ are the only critical numbers in $[0, 2\pi]$.

$f\left(\frac{7\pi}{6}\right) = \dfrac{\frac{-\sqrt{3}}{2}}{2 + \frac{-1}{2}} = \dfrac{-\sqrt{3}}{3}$

$f\left(\frac{11\pi}{6}\right) = \dfrac{\frac{\sqrt{3}}{2}}{2 + \frac{-1}{2}} = \dfrac{\sqrt{3}}{3}$

② $f(0) = \dfrac{1}{2+0} = \dfrac{1}{2} < \dfrac{\sqrt{3}}{3}$

$f(2\pi) = \dfrac{1}{2+0} = \dfrac{1}{2} > \dfrac{-\sqrt{3}}{3}$

③ absolute maximum value is $\dfrac{\sqrt{3}}{3}$

absolute minimum value is $\dfrac{-\sqrt{3}}{3}$

2. Evaluate the following limit using algebraic techniques. (8 points)

$$\lim_{x\to\infty}\left(\sqrt{x^4+6x^2}-x^2\right)$$

$$\frac{\sqrt{x^4+6x^2}-x^2}{1}\cdot\frac{\sqrt{x^4+6x^2}+x^2}{\sqrt{x^4+6x^2}+x^2}=\frac{x^4+6x^2-x^4}{\sqrt{x^4+6x^2}+x^2}$$

$$\lim_{x\to\infty}\left[\sqrt{x^4+6x^2}-x^2\right]=\lim_{x\to 0}\left[\frac{6x^2}{\sqrt{x^4+6x^2}+x^2}\right]$$

$$\frac{\frac{6x^2}{x^2}}{\frac{\sqrt{x^4+6x^2}}{x^2}+\frac{x^2}{x^2}}=\frac{6}{\sqrt{\frac{x^4+6x^2}{x^4}}+1}=\frac{6}{\sqrt{1+\frac{6}{x^2}}+1}$$

$$\lim_{x\to\infty}\left[\sqrt{x^4+6x^2}-x^2\right]=\lim_{x\to\infty}\left[\frac{6}{\sqrt{1+\frac{6}{x^2}}+1}\right]=\frac{6}{\sqrt{1+0}+1}$$

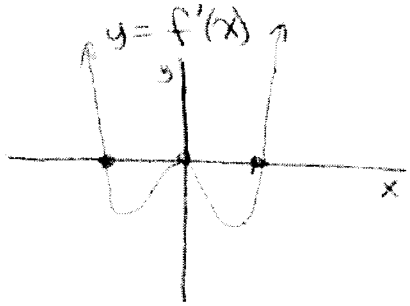
$$=\frac{6}{2}$$

$$=\boxed{3}$$

3. Given that $f(x) = 3x^5 - 5x^3 + 3$, find the following using the principles of calculus. Express your final answers in exact form which means **do not** use decimals

a. Intervals where f is increasing AND where f is decreasing. (4 points)

$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2-1) = 15x^2(x+1)(x-1)$



f is increasing when $f'(x) > 0$.

This happens when $x < -1$ or $x > 1 \Rightarrow (-\infty, -1) \cup (1, \infty)$

f is decreasing when $f'(x) < 0$.

This happens when $-1 < x < 0$ or $0 < x < 1 \Rightarrow (-1, 0) \cup (0, 1)$

b. Coordinates of <u>ALL</u> local maximum and local minimum points. (4 points)

Set $f'(x) = 15x^2(x+1)(x-1) = 0$ Now $f''(x) = 60x^3 - 30x$

critical numbers are $0, -1, 1$

$f(0) = 3$ local maximum at $(-1, 5)$ since $f''(-1) = -30 < 0$

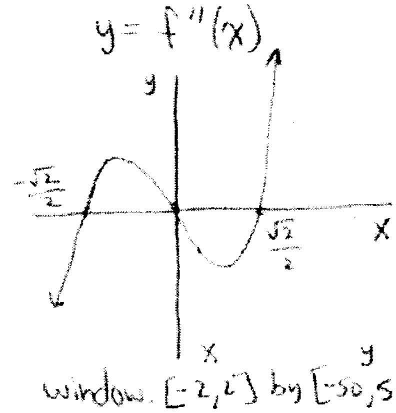
$f(-1) = 5$ local minimum at $(1, 1)$ since $f''(1) = 30 > 0$

$f(1) = 1$

$(0, 3)$ is neither a min. nor a max since $f''(0) = 0$

c. Intervals where f is concave-up and where f is concave-down. (4 points)

$f''(x) = 60x^3 - 30x$



f is concave-up when $f''(x) > 0$

This happens on $\boxed{\left(-\frac{\sqrt{2}}{2}, 0\right) \cup \left(\frac{\sqrt{2}}{2}, \infty\right)}$

f is concave-down when $f''(x) < 0$

This happens on $\boxed{\left(-\infty, -\frac{\sqrt{2}}{2}\right) \cup \left(0, \frac{\sqrt{2}}{2}\right)}$

x –

d. Coordinates of the inflection points. (3 points)

f changes concavity at $x = -\frac{\sqrt{2}}{2}$

$x = 0$

and

$x = \frac{\sqrt{2}}{2}$

4.

4. The function $p(x) = x^3 - 3x^2 + 3x + 1$ is a polynomial. We know that all polynomial functions are continuous and differentiable everywhere. In particular, $p(x)$ is continuous on the closed interval $[-1,5]$ and differentiable on the open interval $(-1,5)$. This satisfies the hypotheses of the Mean Value Theorem.

a. Using function notation, write down the conclusion of the Mean Value Theorem, as learned in class, for this example. (2 points)

There is a number c in (-1,5) such that:

$$p'(c) = \frac{p(5) - p(-1)}{5 - -1}$$

b. For this particular example, there is only one value of c that satisfies the conclusion of the Mean Value Theorem. Find it. (8 points)

$$p'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$$

$$\frac{p(5) - p(-1)}{5 - -1} = \frac{66 - -6}{6} = \frac{72}{6} = 12$$

Solve:

$$p'(c) = 3c^2 - 6c + 3 = 12$$

$$3c^2 - 6c - 9 = 0$$

$$c^2 - 2c - 3 = 0$$

$$(c-3)(c+1) = 0$$

$$c = 3 \text{ or } c = -1$$

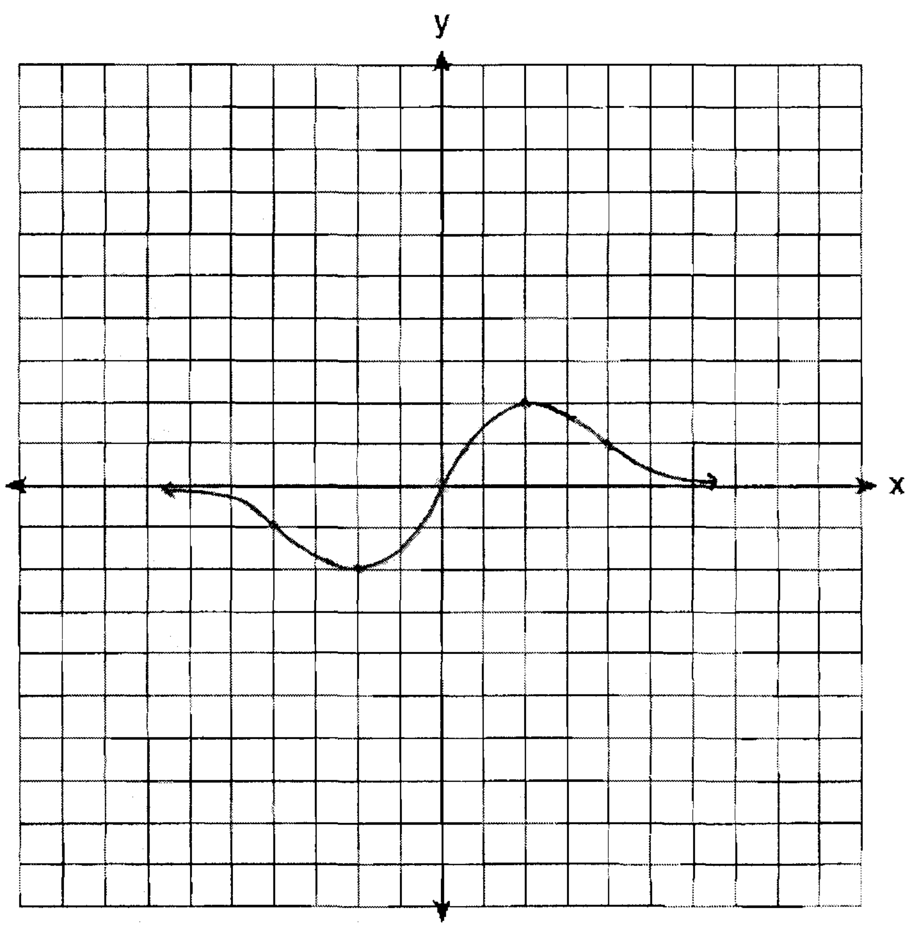
The conclusion states that c must lie in (-1,5)

so ONLY $\boxed{c = 3}$ will work.

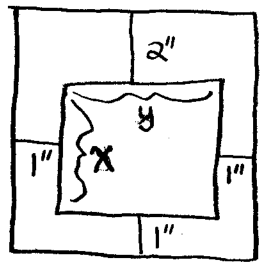
5.

Sketch the graph of a function that satisfies the given conditions. (10 points)

- $f(0) = 0$
- $f'(2) = 0$
- $f'(x) > 0$ if $0 < x < 2$
- $f'(x) < 0$ if $x > 2$
- $f''(x) < 0$ if $0 < x < 4$
- $f''(x) > 0$ if $x > 4$
- $\lim_{x \to \infty} f(x) = 0$
- $f(-x) = -f(x)$ for all values of x.



6. A poster is to have an area of $180\,in^2$ with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions, for the poster, will give the largest printed area? Include a sign chart as part of your response. (10 points)



Let x and y be the dimensions of the Printed area of the poster. This means that the dimensions of the poster itself are $(x+2)$ and $(y+3)$.

$$(x+2)(y+3)=180 \Rightarrow y=\frac{180}{x+2}-3$$

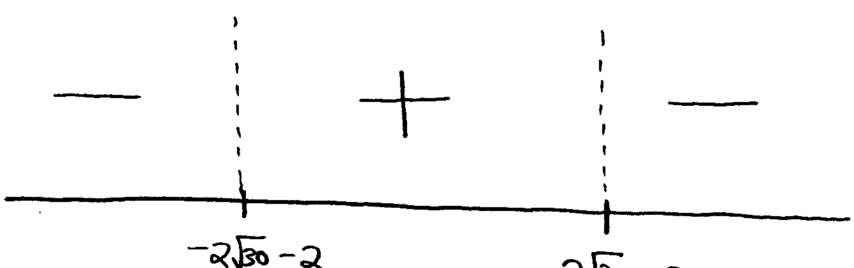
area to be optimized

$$A = x\cdot y = x\left(\frac{180}{x+2}-3\right) = \frac{180x}{x+2}-3x$$

$$A'(x)=\frac{180(x+2)-180x(1)}{(x+2)^2}-3=\frac{360}{(x+2)^2}-3=0 \Rightarrow (x+2)^2=120$$

$$\Rightarrow x=\pm 2\sqrt{30}-2$$

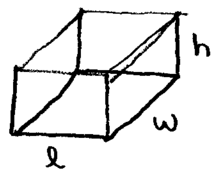
$$\Rightarrow x\approx 8.95\ in^2$$



Since the derivative changes sign at $2\sqrt{30}-2$ from positive to negative, this critical # yields a local maximum.

The dimensions, for the poster, are $2\sqrt{30}$ by $\frac{90}{\sqrt{30}}$ inches.

7. A rectangular storage container with an open top is to have a volume of $10\,m^3$. The length of its base is twice the width. Materials for the bottom of the box cost \$10 per square meter because it is reinforced and hence costs more. Materials for the sides of the box are cheaper, costing only \$6 per square meter. Find the cost of materials for the cheapest such container. Express your critical number in EXACT form for full credit. Do not round or use decimals for the critical number. You may however, round your final answer to the nearest cent. Include a sign chart as part of your response. (10 points)



$\ell = 2w$

$V = \ell \cdot w \cdot h = 2w^2 h = 10$

$\Rightarrow h = \dfrac{5}{w^2}$

constraint equations

We are to optimize Cost.

$$C = 10(\ell \cdot w) + 6(2wh) + 6(2\ell h)$$

$$= 10(2w^2) + 6\left(2w \cdot \frac{5}{w^2}\right) + 6\left(2 \cdot 2w \cdot \frac{5}{w^2}\right)$$

$$= 20w^2 + \frac{60}{w} + \frac{120}{w}$$

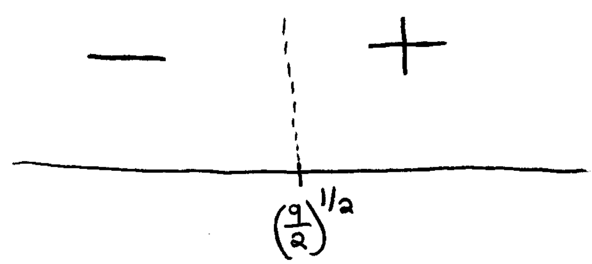
$$C = 20w^2 + \frac{180}{w}$$

$$C'(w) = 40w - \frac{180}{w^2} = 0$$

$$\Rightarrow w^3 = \frac{9}{2}$$

$$\Rightarrow w = \left(\frac{9}{2}\right)^{1/3} \approx 1.65 \text{ meters}$$

$$C\left(\left(\tfrac{9}{2}\right)^{1/3}\right) \approx \boxed{\$\,163.54}$$



Since the derivative changes sign from neg. to pos. at $x = \left(\frac{9}{2}\right)^{1/2}$, this critical # yields a local minimum.

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