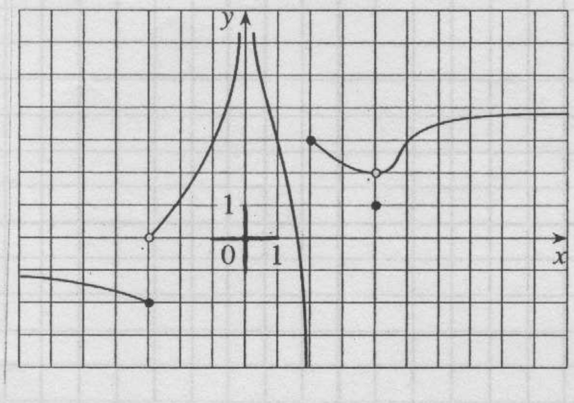
MAC 2311 Calculus with Analytic Geometry I
TEST 1

Name KEY Score 92

1. Use the graph of the function $y = f(x)$ below to find each limit or functional value. If the limit or functional value does not exist or is undefined, so state. (8 points)



$\lim_{x\to 2^+} f(x) =$ 3 $\quad \lim_{x\to 2^-} f(x) =$ $-\infty$ $\quad \lim_{x\to 4} f(x) =$ 2 $\quad \lim_{x\to -3} f(x) =$ DNE

$\lim_{x\to -3^+} f(x) =$ 0 $\quad \lim_{x\to -3^-} f(x) =$ -2 $\quad \lim_{x\to 0} f(x) =$ ∞ $\quad f(4) =$ 1

2. The slope of the tangent line to the graph of the function f at the point $(-3,3)$ is given by the equation $m_{\tan} = \lim_{x\to -3} \dfrac{f(x) - f(-3)}{x+3}$.

a. Use this equation to find the slope of the tangent line to the parabola $f(x) = x^2 + 2x$ at the indicted point. (8points)

$$m_{tan} = \lim_{x\to -3} \frac{f(x)-f(-3)}{x+3}$$

$$= \lim_{x\to -3} \frac{x^2+2x-3}{x+3}$$

$$= \lim_{x\to -3} \frac{(x+3)(x-1)}{x+3}$$

$$= \lim_{x\to -3} (x-1)$$

$$= -3-1$$ by direct substitution prop.

$$= -4$$

b. Find the equation of the tangent line in part a. (2 points)

$$y - y_1 = m(x - x_1)$$

$$y - 3 = -4(x - -3)$$

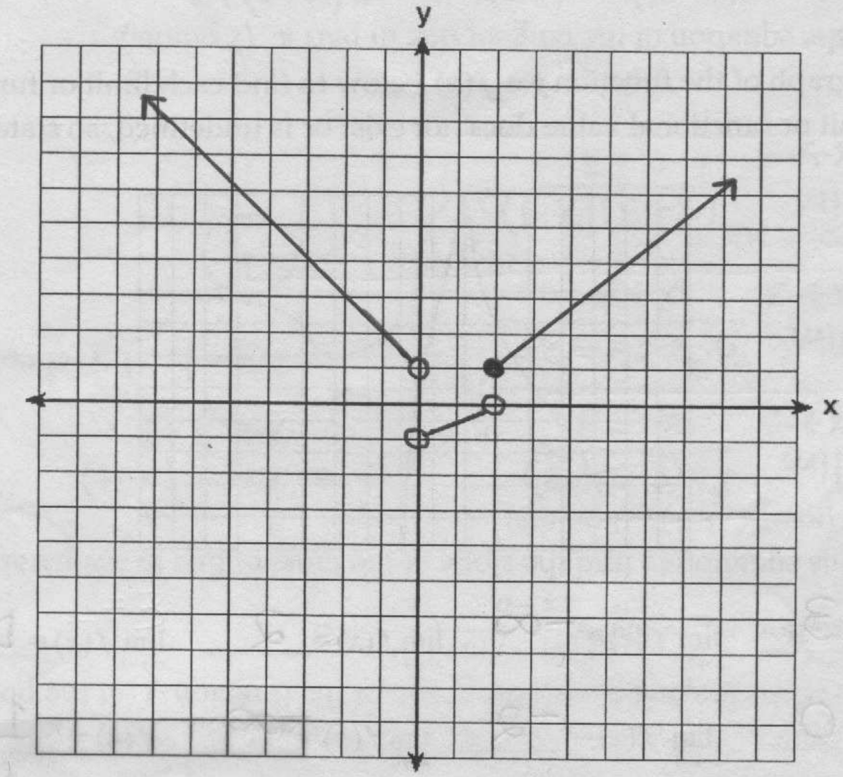
$$y = -4(x+3)+3$$

$$y = -4x - 12 + 3$$

$$y = -4x - 9$$

3. Sketch the graph of a function that satisfies all of the conditions below. (8 points)

$\lim_{x\to 0^-} f(x)=1,\quad \lim_{x\to 0^+} f(x)=-1,\quad \lim_{x\to 2^-} f(x)=0,\quad \lim_{x\to 2^+} f(x)=1,\quad f(2)=1,\quad f(0)$ is undefined



This is ONE Possible graph

4. Compute the following limit, if it exists. Use your graphing calculator to check that your answer makes sense. (6 points)

$$\lim_{x\to 1}\left(\frac{x^2+x-2}{x^2-1}\right)$$

$$= \lim_{x\to 1}\left[\frac{(x+2)(x-1)}{(x+1)(x-1)}\right]$$

$$= \lim_{x\to 1}\left[\frac{x+2}{x+1}\right]$$

$$= \frac{1+2}{1+1} = \frac{3}{2}$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(4-x^2) = 4-(2)^2 = 4-4=0$$

5. Let $f(x) = \begin{cases} 4-x^2 & \text{if } x \le 2 \\ x-1 & \text{if } x > 2 \end{cases}$

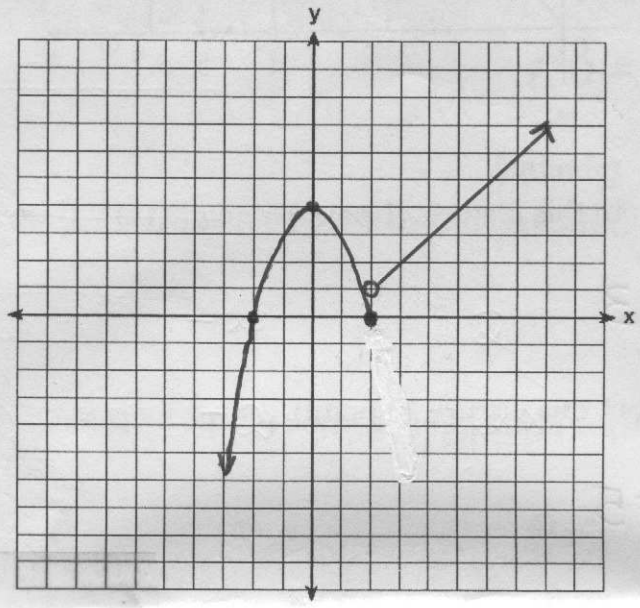
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x-1) = 2-1 = 1$$

a. $\lim_{x\to 2^-} f(x) =$ <u>0</u> and $\lim_{x\to 2^+} f(x) =$ <u>1</u> (2points)

b. Does $\lim_{x\to 2} f(x)$ exist? Why or why not? (~~1~~ 2 points)

$\lim_{x\to 2} f(x)$ <u>DOES</u> <u>NOT</u> exist b/c $0 = \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x) = 1$

c. Sketch the graph of this function below. (3 points)



6. Compute the following limit, if it exists. Use your graphing calculator to make sure your answer makes sense. (8 points)

$$\lim_{x\to 10}\left(\frac{\sqrt{x+6}-4}{x-10}\right)$$

$$= \lim_{x\to 10} \frac{\sqrt{x+6}-4}{x-10} \cdot \frac{\sqrt{x+6}+4}{\sqrt{x+6}+4}$$

$$= \lim_{x\to 10} \frac{x+6-16}{(x-10)[\sqrt{x+6}+4)} = \lim_{x\to 10} \frac{x-10}{(x-10)[\sqrt{x+6}+4]}$$

$$= \lim_{x\to 10} \frac{1}{\sqrt{x+6}+4}$$

$$\overset{D.S.P}{=} \frac{1}{\quad} = \frac{1}{\quad} = \boxed{\frac{1}{\quad}}$$

7. A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in gallons) after t minutes.

t (min.)	5	10	15	20	25	30
V (gal.)	694	444	250	111	28	0

Part a) If P is the point (15, 250) on the graph of V, find the slope of the secant line PQ where Q is the point on the graph when $t = 25$. (2 points)

$$m_{sec} = \frac{250-28}{15-25} = \frac{-111}{5} = \boxed{-22.2}$$

Part b) Give the best estimate for the slope of the tangent line at P by averaging the slopes of two secant lines. (3 points)

$$m_{tan} \approx \left(\frac{444-250}{10-15} + \frac{250-111}{15-20}\right) \Big/ 2 = \boxed{-33.3}$$

Part c) Write a sentence or two describing what the equation $f'(15) = -32$ means in terms of the draining tank. Be as specific as possible by including units of measurement. (2 points)

15 minutes after the tank starts to drain, the amount of water in the tank is decreasing at a rate of 32 gallons per minute.

Part d) According to the table, during what time interval is the tank draining the fastest? (1 point)

The tank is draining the fastest during the first 5-second interval.

Part e) What happens to the rate at which the tank is draining as time passes? Explain why this might happen? (2 point)

As time passes, the rate at which the tank drains slows down. This happens because the weight of the water over the drain lessens as more water is drained out resulting in lower pressure.

8. Use the alternative definition for the slope of the tangent line given by $m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ to find the slope of the tangent line to the curve $y = \frac{2}{x+3}$ at the point $(a, f(a))$. (8 points)

$$m_{tan} = \lim_{h \to 0} \left[\frac{\frac{2}{a+h+3} - \frac{2}{a+3}}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\frac{2(a+3) - 2(a+h+3)}{(a+3)(a+h+3)}}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{2a+6-2a-2h-6}{(a+3)(a+h+3)} \cdot \frac{1}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{-2h}{h(a+3)(a+h+3)} \right]$$

$$= \lim_{h \to 0} \frac{-2}{(a+3)(a+h+3)}$$

$$= \frac{-2}{(a+3)(a+0+3)} = \frac{-2}{(a+3)(a+3)} = \frac{-2}{(a+3)^2}$$

Find the equation of the line tangent this curve at (-1, 1). (2 points)

The slope when $a=-1$ is $\frac{-2}{(-1+3)^2} = \frac{-2}{2^2} = \frac{-2}{4} = \frac{-1}{2} =$ slope

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{-1}{2}(x - -1)$$

$$y - 1 = \frac{-1}{2}(x+1)$$

$$y - 1 = \frac{-1}{2}x - \frac{1}{2}$$

$$y = \frac{-1}{2}x - \frac{1}{2} + 1$$

$$\boxed{y = \frac{-1}{2}x + \frac{1}{2}}$$

9. As given in class, the Intermediate Value Theorem (IVT) has three conditions in its hypothesis. Write down all three in the blanks provided.

f is continuous on [a,b] (1 point)

N is any number between f(a) and f(b) (1 point)

$f(a) \neq f(b)$ (1 point)

Write in a sentence the conclusion of the IVT in the space below.

There exists at least one number c in (a,b) such that f(c)=N. (2 points)

If $f(x) = x^3 - 9x^2 + 23x$ then we can use the IVT to show that there is a number c such that $f(c) = 15$. We will outline how this is to be done below.

I. Find two distinct numbers a and b such that $f(b) > 15$ and $f(a) < 15$ (2 points)

when $b=2$, $f(b)=18>15$

when $a=0$, $f(a)=0<15$

II. $f(x)$ is a polynomial function. What special property do ALL polynomial functions enjoy on ANY closed interval that relates to the IVT? (1 point)

Continuity

III. Use algebra or your graphing calculator to find a number c such that $f(c) = 15$. (1 points)

c must be between 0 and 2.

Solve $f(c)-15=0$ or $x^3-9x^2+23x-15=0$

$(x-1)(x-3)(x-5)=0$

$x=1,3,5$ The c that we want is $\boxed{c=1}$

★ Note: There is more than one way to do this part that depends on your choice of a and b.

10. Compute the limit, if it exists. (8 points)

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$$

$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h}$$

$$= \lim_{h \to 0} \frac{6h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(6+h)}{h}$$

$$= \lim_{h \to 0} (6+h)$$

$$\overset{\text{D.S.P}}{=} 6 + 0$$

$$= 6$$

11. Find the constant c that makes g a continuous function on $(-\infty, \infty)$. (8 points)

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4 \end{cases}$$

The only "suspicious" point we need to check continuity at is $x = 4$ since $x^2 - c^2$ and $cx + 20$ are both polynomials and are continuous everywhere else.

To this end, we need to make sure $\lim_{x \to 4} g(x) = g(4)$ (Defn. of continuity)

This boils down to making sure $\lim_{x \to 4} g(x) = 4c + 20$, since $g(4) = 4c + 20$.

For $\lim_{x \to 4} g(x)$ to exist, we need $\lim_{x \to 4^-} g(x) = \lim_{x \to 4^+} g(x)$. Now, $\lim_{x \to 4^-} g(x) = \lim_{x \to 4^-} (x^2 - c^2)$
$= (4)^2 - c^2$
$= 16 - c^2$

Also, $\lim_{x \to 4^+} g(x) = \lim_{x \to 4^+} (cx + 20) = 4c + 20$

Thus, it suffices to solve the equation $4c + 20 = 16 - c^2$.

or

$c^2 + 4c + 4 = 0$

$(c+2)^2 = 0$

$c + 2 = 0$

$c = -2$

It can readily be checked that $\lim_{x \to 4} g(x) = g(4)$ when $c = -2$.

Hence, g is continuous on $(-\infty, \infty)$ when $c = -2$