MAC 2311 Calculus with Analytic Geometry I

TEST 3

Name KEY Score $\overline{75}$

1. Use calculus and algebra to find ALL critical numbers of the function $f(x) = x^{4/5}(x-4)^2$ (8 points)

$$f'(x) = \frac{4}{5}x^{-1/5} \cdot (x-4)^2 + x^{4/5} \cdot 2(x-4)^{2-1} \cdot (1)$$

$$f'(x) = \frac{4(x-4)^2}{5x^{1/5}} + \frac{2x^{4/5}(x-4)}{1}$$

$$f'(x) = \frac{4(x-4)^2 + 10x(x-4)}{5x^{1/5}}$$

$f'(x) = 0$ when $4(x-4)^2 + 10x(x-4) = 0$

$$(x-4)\left[4(x-4) + 10x\right] = 0$$

$$(x-4)(14x-16) = 0$$

$$x = 4 \text{ OR } x = \frac{8}{7}$$

$f'(x)$ is undefined when $5x^{1/5} = 0$

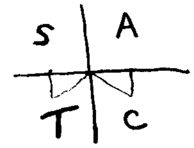
$$x^{1/5} = 0$$

$$x = 0$$

The critical numbers are $0, \frac{8}{7},$ and $4$.

2. Use the closed interval method to find the absolute maximum and minimum values of $f(x) = x - 2\cos x$ on the interval $[-\pi, \pi]$. Be sure to include all steps of this method. Express your answer in exact form. DO NOT use decimal approximations. (10 points)

$f'(x) = 1 + 2\sin x = 0$




$\theta_r = \frac{\pi}{6}$

$\sin x = \frac{-1}{2}$

$x = -\pi + \frac{\pi}{6}$ or $x = -\frac{\pi}{6}$

$x = \frac{-5\pi}{6}$ or $x = -\frac{\pi}{6}$ critical numbers

$f\left(\frac{-5\pi}{6}\right) = \frac{-5\pi}{6} - 2\cos\left(\frac{-5\pi}{6}\right) = \frac{-5\pi}{6} - 2\left(\frac{-\sqrt{3}}{2}\right) = \frac{-5\pi}{6} + \sqrt{3}$ ①

$f\left(\frac{-\pi}{6}\right) = \frac{-\pi}{6} - 2\cos\left(\frac{-\pi}{6}\right) = \frac{-\pi}{6} - 2\left(\frac{\sqrt{3}}{2}\right) = \frac{-\pi}{6} - \sqrt{3}$ ②

$f(-\pi) = -\pi - 2\cos(-\pi) = -\pi - 2(-1) = -\pi + 2$ ③

$f(\pi) = \pi - 2\cos(\pi) = \pi - 2\cos(-1) = \pi + 2$ ④

Abs. max. is $\pi + 2$ and occurs at $x = \pi$.

Abs. min. is $\frac{-\pi}{6} - \sqrt{3}$ and occurs at $x = -\frac{\pi}{6}$

3. The function $g(x)=\dfrac{x}{x+2}$ is continuous on the closed interval $[1,4]$ and differentiable on the open interval $(1,4)$. These conditions satisfy the hypotheses of the Mean Value Theorem with respect to this function. Using calculus and algebra, find all numbers $c$ that satisfy the conclusion to the theorem. (10 points)

$$g'(c)=\frac{f(4)-f(1)}{4-1}=\frac{\frac{4}{6}-\frac{1}{3}}{3}=\frac{\frac{2}{6}}{\frac{3}{1}}=\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{9}$$

$$g'(x)=\frac{(x+2)(1)-(x)(1)}{(x+2)^2}=\frac{x+2-x}{(x+2)^2}=\frac{2}{(x+2)^2}$$

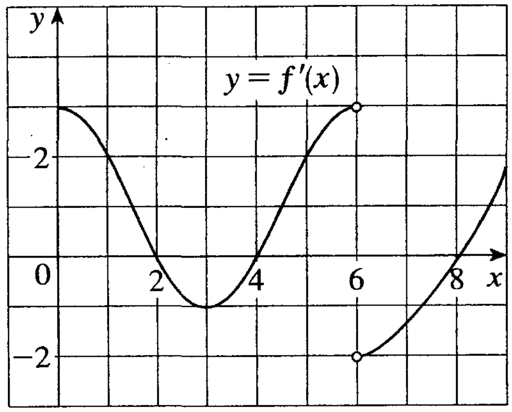
The conclusion of the MVT states that there is a number $c$ in $(1,4)$ with $\dfrac{2}{(c+2)^2}=\dfrac{1}{9}$. This implies $(c+2)^2=18$

$$c+2=\pm\sqrt{18}$$

$$c=-2\pm\sqrt{18}$$

but $-2-\sqrt{18}$ is not in the interval $(1,4)$ so only $c=-2+\sqrt{18}$ will work.

4. The graph of the derivative $f'$ of a continuous function $f$ is shown below. Answer the following questions.




a) On what intervals is $f$ increasing or decreasing? (3 points)

$f$ is inc. on $(0,2) \cup (4,6) \cup (8,\infty)$

$f$ is dec. on $(2,4) \cup (6,8)$

b) At what values does $f$ have a local maximum or minimum? (3 points)

$f$ has a local maximum at $x = 2, 6$

$f$ has a local minimum at $x = 4, 8$

c) On what intervals is $f$ concave upward or downward? (2 points)

$f$ is concave upward on $(3,\infty)$ or $(3,6) \cup (6,\infty)$

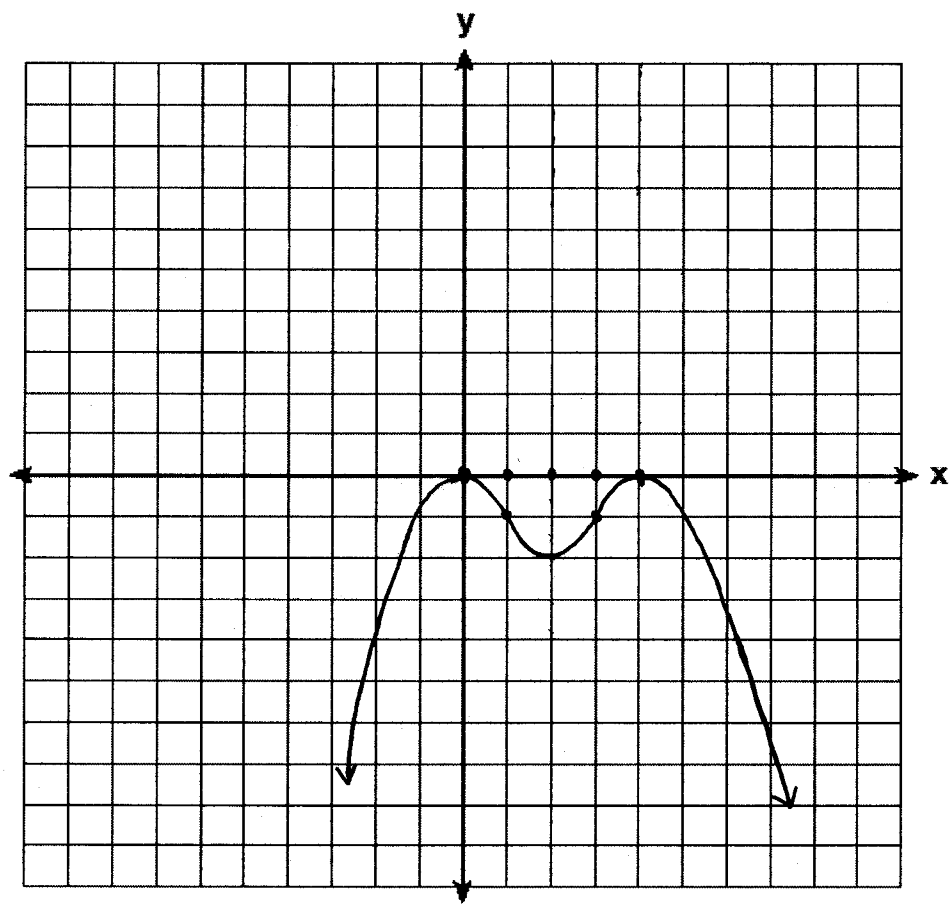
$f$ is concave downward on $(0,3)$

d) What are the $x$-values of the inflection point(s)? (1 points)

$x = 3$ gives the only inflection pt.

5. Sketch the graph of a function that satisfies all of the given conditions. (10 points)

- $f'(0) = f'(2) = f'(4) = 0$
- $f'(x) > 0$ if $x < 0$ or $2 < x < 4$
- $f'(x) < 0$ if $0 < x < 2$ or $x > 4$
- $f''(x) > 0$ if $1 < x < 3$
- $f''(x) < 0$ if $x < 1$ or $x > 3$



6. Use <u>algebraic</u> techniques to compute the following limit. You must show <u>ALL</u> work for full credit. (8 points)

$$\lim_{x\to\infty}\left(\sqrt{x^4+6x^2}-x^2\right)$$

$$\frac{\sqrt{x^4+6x^2}-x^2}{1}\cdot\frac{\sqrt{x^4+6x^2}+x^2}{\sqrt{x^4+6x^2}+x^2}=\frac{x^4+6x^2-x^4}{\sqrt{x^4+6x^2}+x^2}=\frac{6x^2}{\sqrt{x^4+6x^2}+x^2}$$

divide num. and den. by $x^2$. Since $x\to\infty$, $x^2=\sqrt{x^4}$ and so we get

$$\frac{\frac{6x^2}{x^2}}{\frac{\sqrt{x^4+6x^2}}{\sqrt{x^4}}+\frac{x^2}{x^2}}=\frac{6}{\sqrt{1+\frac{6}{x^2}}+1}$$

This means $\lim\limits_{x\to\infty}\left(\sqrt{x^4+6x^2}-x^2\right)=\lim\limits_{x\to\infty}\dfrac{6}{\sqrt{1+\frac{6}{x^2}}+1}=\dfrac{6}{\sqrt{1+0}+1}=\dfrac{6}{2}$

$$=\boxed{3}$$

7. A metal can company has an order to make cylindrical cans with a volume of $250\,cm^3$. What should be the dimensions of the can be in order to use the least amount of metal in their production? (Hint: the volume formula for a cylindrical can is $V = \pi r^2 h$ while the total surface area is $S = 2\pi r^2 + 2\pi r h$) (10 points)

$$V = \pi r^2 h \rightarrow 250 = \pi r^2 h \rightarrow h = \frac{250}{\pi r^2}$$

$$S = 2\pi r^2 + 2\pi r h \rightarrow 2\pi r^2 + 2\pi r\left(\frac{250}{\pi r^2}\right) \rightarrow S = 2\pi r^2 + \frac{500}{r}$$

$$S' = 4\pi r - \frac{500}{r^2} = 0 \rightarrow 4\pi r = \frac{500}{r^2} \rightarrow r^3 = \frac{125}{\pi} \rightarrow r = \frac{5}{\sqrt[3]{\pi}} \approx 3.414$$

$\frac{5}{\sqrt[3]{\pi}}$ is the only critical number.

1st derivative test

$$- \quad \Big| \quad +$$

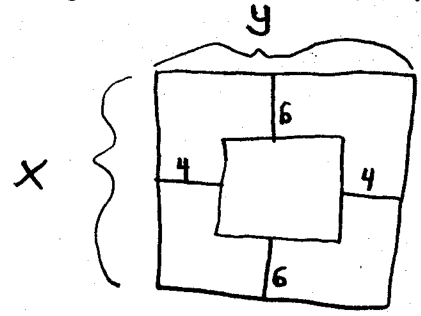
$$\frac{5}{\sqrt[3]{\pi}}$$

Since the 1st derivative changes sign from neg. to pos. at $\frac{5}{\sqrt[3]{\pi}}$ there is a local minimum at this number.

The dimensions are $r = \frac{5}{\sqrt[3]{\pi}}$ and $h = \frac{10}{\sqrt[3]{\pi}}$

8. The top and bottom margins of a poster are each 6 $cm$ and the side margins are each 4 $cm$. If the area of printed material on the poster is fixed at 384 $cm^2$, find the dimensions of the poster with smallest area. (10 points)

Let $x$ and $y$ be the dimensions of the poster.



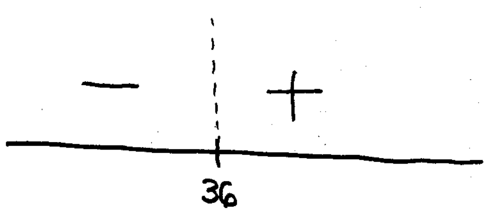

$$(x-12)(y-8)=384 \rightarrow y=\frac{384}{x-12}+8$$

$$A=xy=x\left(\frac{384}{x-12}+8\right)=\frac{384x}{x-12}+8x$$

$$A'=\frac{(x-12)(384)-(384x)(1)}{(x-12)^2}+8=0$$

$$A'=\frac{-4608}{(x-12)^2}+8=0 \rightarrow \frac{4608}{(x-12)^2}=8 \rightarrow (x-12)^2=576$$

$$\rightarrow \boxed{\begin{array}{l} x=36\text{ cm} \\ y=24\text{ cm} \end{array}}$$




Since the derivative changes sign from neg. to pos. at $x=36$, there is a local minimum there.