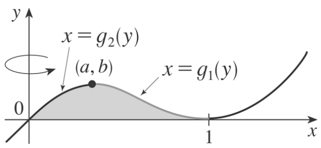
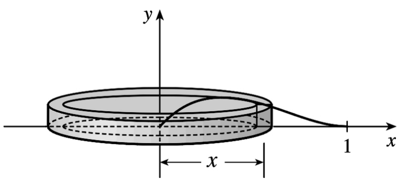
1.




If we were to use the "washer" method, we would first have to locate the local maximum point (a,b) *of* $y=x(x-1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y=x(x-1)^2$ for $x$ in terms of $y$ to obtain the functions $x=g_1(y)$ and $x=g_2(y)$ shown in the first figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using $V=\pi\int_0^b \left\{ \left[g_1(y)\right]^2-\left[g_2(y)\right]^2 \right\} dy$ .

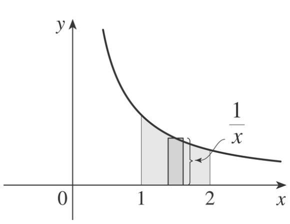
Using shells, we find that a typical approximating shell has radius $x$ , so its circumference is $2\pi x$. Its height is $y$ , that is, $x(x-1)^2$. So the total volume is

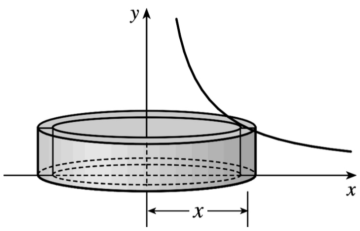
$$V=\int_0^1 2\pi x\left[x(x-1)^2\right]dx=2\pi\int_0^1 \left(x^4-2x^3+x^2\right)dx=2\pi\left[\frac{x^5}{5}-2\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1=\frac{\pi}{15}$$

3.

$$V=\int_1^2 2\pi x\cdot\frac{1}{x}\,dx=2\pi\int_1^2 1\,dx$$

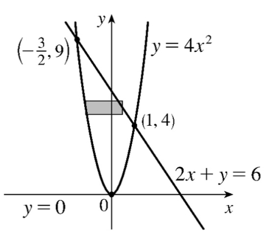
$$=2\pi\left[x\right]_1^2=2\pi(2-1)=2\pi$$

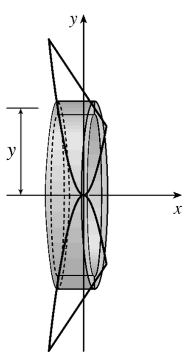





13. The curves intersect when $4x^2=6-2x \Leftrightarrow 2x^2+x-3=0 \Leftrightarrow (2x+3)(x-1)=0 \Leftrightarrow x=-\frac{3}{2}$ or 1. Solving the equations for $x$ gives us $y=4x^2 \Rightarrow x=\pm\frac{1}{2}\sqrt{y}$ and $2x+y=6 \Rightarrow x=-\frac{1}{2}y+3$ .

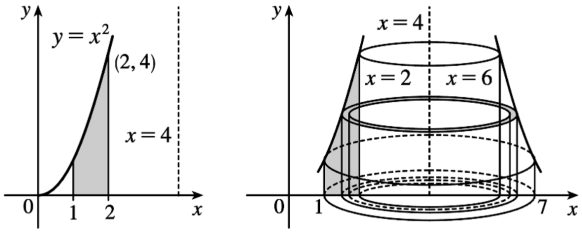
$$
\begin{aligned}
V &= 2\pi\int_0^4\left\{y\left[\left(\frac{1}{2}\sqrt{y}\right)-\left(-\frac{1}{2}\sqrt{y}\right)\right]\right\}dy+2\pi\int_4^9\left\{y\left[\left(-\frac{1}{2}y+3\right)-\left(-\frac{1}{2}\sqrt{y}\right)\right]\right\}dy \\
&= 2\pi\int_0^4\left(y\sqrt{y}\right)dy+2\pi\int_4^9\left(-\frac{1}{2}y^2+3y+\frac{1}{2}y^{3/2}\right)dy=2\pi\left[\frac{2}{5}y^{5/2}\right]_0^4+2\pi\left[-\frac{1}{6}y^3+\frac{3}{2}y^2+\frac{1}{5}y^{5/2}\right]_4^9 \\
&= 2\pi\left(\frac{2}{5}\cdot 32\right)+2\pi\left[\left(-\frac{243}{2}+\frac{243}{2}+\frac{243}{5}\right)-\left(-\frac{32}{3}+24+\frac{32}{5}\right)\right] \\
&= \frac{128}{5}\pi+2\pi\left(\frac{433}{15}\right)=\frac{1250}{15}\pi=\frac{250}{3}\pi
\end{aligned}
$$



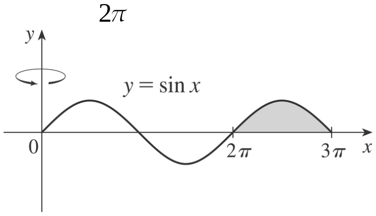




17.

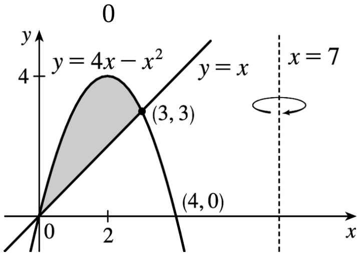
$$
\begin{aligned}
V &= \int_1^2 2\pi(4-x)x^2\,dx=2\pi\left[\frac{4}{3}x^3-\frac{1}{4}x^4\right]_1^2 \\
&= 2\pi\left[\left(\frac{32}{3}-4\right)-\left(\frac{4}{3}-\frac{1}{4}\right)\right]=\frac{67}{6}\pi
\end{aligned}
$$




21. $V = \int_{2\pi}^{3\pi} 2\pi x \sin x \, dx$




22. $V = \int_{0}^{3} 2\pi \, (7-x) \left[ \left(4x-x^2\right) - x \right] dx$

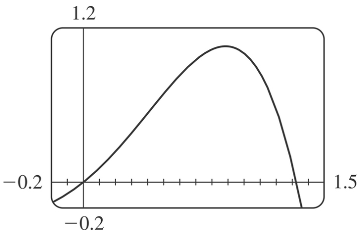



29. $\int_{0}^{3} 2\pi x^5 \, dx = 2\pi \int_{0}^{3} x(x^4) \, dx$ . The solid is obtained by rotating the region $0 \le y \le x^4$ , $0 \le x \le 3$ about the $y$ –axis using cylindrical shells.

31. $\int_{0}^{1} 2\pi (3-y)(1-y^2) \, dy$ . The solid is obtained by rotating the region bounded by (i) $x = 1 - y^2$ , $x = 0$ , and $y = 0$ or (ii) $x = y^2$ , $x = 1$ , and $y = 0$ about the line $y = 3$ using cylindrical shells.

32. $\int_{0}^{\pi/4} 2\pi (\pi - x)(\cos x - \sin x) \, dx$ . The solid is obtained by rotating the region bounded by (i) $0 \le y \le \cos x - \sin x$ , $0 \le x \le \frac{\pi}{4}$ or (ii) $\sin x \le y \le \cos x$ , $0 \le x \le \frac{\pi}{4}$ about the line $x = \pi$ using cylindrical shells.

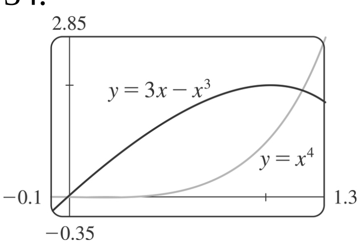
33.




From the graph, the curves intersect at $x=0$ and at $x=a\approx 1.32$ , with $x+x^2-x^4>0$ on the interval $(0,a)$. So the volume of the solid obtained by rotating the region about the $y-$ axis is

$$V = 2\pi\int_0^a \left[x(x+x^2-x^4)\right]dx=2\pi\int_0^a (x^2+x^3-x^5)dx$$
$$= 2\pi\left[\frac{1}{3}x^3+\frac{1}{4}x^4-\frac{1}{6}x^6\right]_0^a\approx 4.05$$

34.

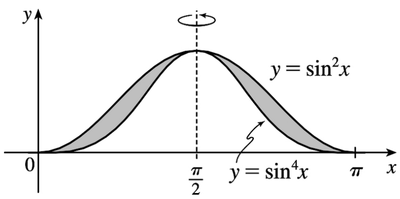



From the graph, the curves intersect at $x=0$ and at $x=a\approx 1.17$ , with $3x-x^3>x^4$ on the interval $(0,a)$. So the volume of the solid obtained by rotating the region about the $y-$ axis is

$$V = 2\pi\int_0^a \left\{x[(3x-x^3)-x^4]\right\} dx=2\pi\int_0^a (3x^2-x^4-x^5)dx$$
$$= 2\pi\left[x^3-\frac{1}{5}x^5-\frac{1}{6}x^6\right]_0^a\approx 4.62$$

35.

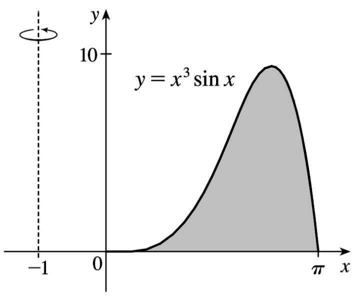
$$V = 2\pi\int_0^{\pi/2}\left[\left(\frac{\pi}{2}-x\right)\left(\sin^2 x-\sin^4 x\right)\right]dx$$
$$\overset{\text{CAS}}{=} \frac{1}{32}\pi^3$$




36.

$$V = 2\pi \int_0^{\pi} \left\{ [x-(-1)](x^3 \sin x) \right\} dx \overset{\text{CAS}}{=} 2\pi(\pi^4+\pi^3-12\pi^2-6\pi+48)$$

$$= 2\pi^5+2\pi^4-24\pi^3-12\pi^2+96\pi$$




37. Use disks:

$$V = \int_{-2}^{1} \pi \left(x^2+x-2\right)^2 dx = \pi \int_{-2}^{1} \left(x^4+2x^3-3x^2-4x+4\right) dx$$

$$= \pi \left[ \frac{1}{5} x^5 + \frac{1}{2} x^4 - x^3 - 2x^2 + 4x \right]_{-2}^{1} = \pi \left[ \left( \frac{1}{5} + \frac{1}{2} - 1 - 2 + 4 \right) - \left( -\frac{32}{5} + 8 + 8 - 8 - 8 \right) \right]$$

$$= \pi \left( \frac{33}{5} + \frac{3}{2} \right) = \frac{81}{10} \pi$$

38. Use shells:

$$V = \int_1^2 2\pi x \left(-x^2+3x-2\right) dx = 2\pi \int_1^2 \left(-x^3+3x^2-2x\right) dx$$

$$= 2\pi \left[ -\frac{1}{4} x^4 + x^3 - x^2 \right]_1^2 = 2\pi \left[ (-4+8-4) - \left( -\frac{1}{4} + 1 - 1 \right) \right] = \frac{\pi}{2}$$

39. Use shells:

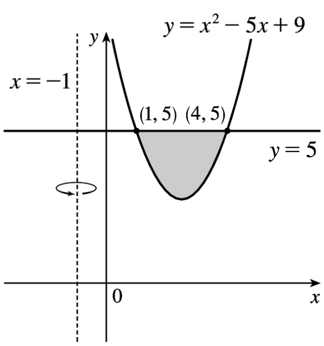
$$V = \int_1^4 2\pi [x-(-1)]\left[5-\left(x^2-5x+9\right)\right]dx$$

$$= 2\pi \int_1^4 (x+1)\left(-x^2+5x-4\right)dx$$

$$= 2\pi \int_1^4 \left(-x^3+4x^2+x-4\right)dx = 2\pi\left[-\frac{1}{4}x^4+\frac{4}{3}x^3+\frac{1}{2}x^2-4x\right]_1^4$$

$$= 2\pi\left[\left(-64+\frac{256}{3}+8-16\right)-\left(-\frac{1}{4}+\frac{4}{3}+\frac{1}{2}-4\right)\right]$$

$$= 2\pi\left(\frac{63}{4}\right) = \frac{63\pi}{2}$$



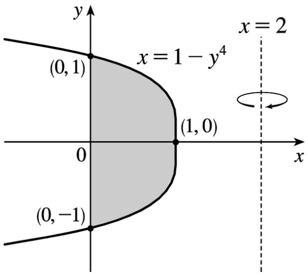

40. Use washers:

$$V = \int_{-1}^{1} \pi \left\{[2-0]^2-\left[2-\left(1-y^4\right)\right]^2\right\} dy$$

$$= 2\pi \int_0^1 \left[4-\left(1+y^4\right)^2\right]dy \text{ [by symmetry]}$$

$$= 2\pi \int_0^1 \left[4-\left(1+2y^4+y^8\right)\right]dy = 2\pi \int_0^1 \left(3-2y^4-y^8\right)dy$$

$$= 2\pi\left[3y-\frac{2}{5}y^5-\frac{1}{9}y^9\right]_0^1 = 2\pi\left(3-\frac{2}{5}-\frac{1}{9}\right) = 2\pi\left(\frac{112}{45}\right) = \frac{224\pi}{45}$$




41. Use disks: $V=\pi\int_0^2 \left[\sqrt{1-(y-1)^2}\right]^2 dy=\pi\int_0^2 \left(2y-y^2\right)dy=\pi\left[y^2-\frac{1}{3}y^3\right]_0^2=\pi\left(4-\frac{8}{3}\right)=\frac{4}{3}\pi$

42. Using shells, we have

$$V=\int_0^2 2\pi y\left[\sqrt{1-(y-1)^2}-\left(-\sqrt{1-(y-1)^2}\right)\right]dy$$

$$=2\pi\int_0^2 y\cdot 2\sqrt{1-(y-1)^2}\,dy=4\pi\int_{-1}^1 (u+1)\sqrt{1-u^2}\,du \quad [\text{let } u=y-1]$$

$$=4\pi\int_{-1}^1 u\sqrt{1-u^2}\,du+4\pi\int_{-1}^1 \sqrt{1-u^2}\,du$$

The first definite integral equals zero because its integrand is an odd function. The second is the area of a semicircle of radius 1 , that is, $\frac{\pi}{2}$ . Thus, $V=4\pi\cdot 0+4\pi\cdot\frac{\pi}{2}=2\pi^2$.

43.

$$V=2\int_0^r 2\pi x\sqrt{r^2-x^2}\,dx=-2\pi\int_0^r \left(r^2-x^2\right)^{1/2}(-2x)\,dx=\left[-2\pi\cdot\frac{2}{3}\left(r^2-x^2\right)^{3/2}\right]_0^r$$

$$=-\frac{4}{3}\pi\left(0-r^3\right)=\frac{4}{3}\pi r^3$$

44.

$$V=\int_{R-r}^{R+r} 2\pi x\cdot 2\sqrt{r^2-(x-R)^2}\,dx$$

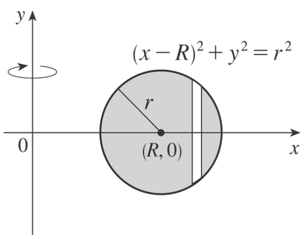
$$=\int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du \qquad [\text{ let } u=x-R\ ]$$

$$=4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du+4\pi\int_{-r}^{r}u\sqrt{r^2-u^2}\,du$$

The first integral is the area of a semicircle of radius $r$ , that is, $\frac{1}{2}\pi r^2$ ,

and the second is zero since the integrand is an odd function. Thus, $V=4\pi R\left(\frac{1}{2}\pi r^2\right)+4\pi\cdot 0=2\pi Rr^2$

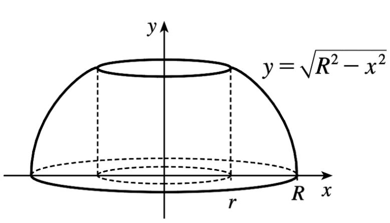
.




45. $V=2\pi\int_0^r x\left(-\frac{h}{r}x+h\right)dx=2\pi h\int_0^r\left(-\frac{x^2}{r}+x\right)dx=2\pi h\left[-\frac{x^3}{3r}+\frac{x^2}{2}\right]_0^r=2\pi h\frac{r^2}{6}=\frac{\pi r^2 h}{3}$

46. By symmetry, the volume of a napkin ring obtained by drilling a hole of radius $r$ through a sphere with radius $R$ is twice the volume obtained by rotating the area above the $x$– axis and below the curve $y=\sqrt{R^2-x^2}$ (the equation of the top half of the cross–section of the sphere), between $x=r$ and $x=R$ , about the $y$– axis.
This volume is equal to

$$2\int_{\text{inner radius}}^{\text{outer radius}}2\pi rh\,dx=2\cdot 2\pi\int_r^R x\sqrt{R^2-x^2}\,dx=4\pi\left[-\frac{1}{3}\left(R^2-x^2\right)^{3/2}\right]_r^R=\frac{4}{3}\pi\left(R^2-r^2\right)^{3/2}$$




But by the Pythagorean Theorem, $R^2-r^2=\left(\frac{1}{2}h\right)^2$ , so the volume of the napkin ring is

$\frac{4}{3}\pi\left(\frac{1}{2}h\right)^3=\frac{1}{6}\pi h^3$ , which is independent of both $R$ and $r$ ; that is, the amount of wood in a napkin ring of height $h$ is the same regardless of the size of the sphere used. Note that most of this calculation has been done already, but with more difficulty, in Exercise 6.2.68.

*Another solution:* The height of the missing cap is the radius of the sphere minus half the height of the cut–out cylinder, that is, $R-\frac{1}{2}h$ . Using Exercise 6.2.49,

$$V_{\text{napkin ring}}=V_{\text{sphere}}-V_{\text{cylinder}}-2V_{\text{cap}}=\frac{4}{3}\pi R^3-\pi r^2 h-2\cdot\frac{\pi}{3}\left(R-\frac{1}{2}h\right)^2\left[3R-\left(R-\frac{1}{2}h\right)\right]=\frac{1}{6}\pi h^3$$