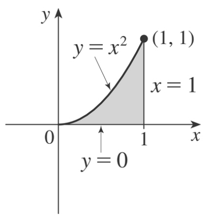
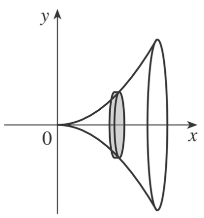
1. A cross–section is circular with radius x^2, so its area is $A(x)=\pi\left(x^2\right)^2$.

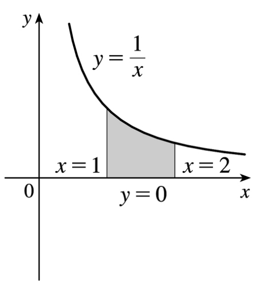
$$V=\int_0^1 A(x)\,dx=\int_0^1 \pi\left(x^2\right)^2 dx=\pi\int_0^1 x^4 dx=\pi\left[\frac{1}{5}x^5\right]_0^1=\frac{\pi}{5}$$

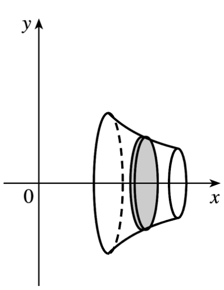




3. A cross–section is a disk with radius $1/x$, so its area is $A(x)=\pi(1/x)^2$.

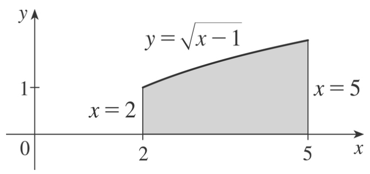
$$V=\int_1^2 A(x)\,dx=\int_1^2 \pi\left(\frac{1}{x}\right)^2 dx=\pi\int_1^2 \frac{1}{x^2}\,dx=\pi\left[-\frac{1}{x}\right]_1^2=\pi\left[-\frac{1}{2}-(-1)\right]=\frac{\pi}{2}$$

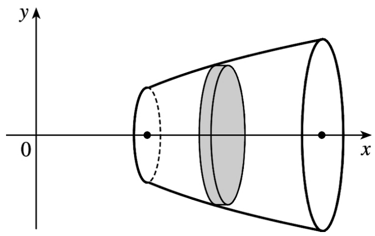




4. A cross–section is circular with radius $\sqrt{x-1}$, so its area is $A(x)=\pi\left(\sqrt{x-1}\right)^2=\pi(x-1)$.

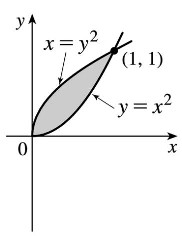
$$V=\int_2^5 A(x)\,dx=\int_2^5 \pi(x-1)\,dx=\pi\left[\frac{1}{2}x^2-x\right]_2^5=\pi\left(\frac{25}{2}-5-\frac{4}{2}+2\right)=\frac{15}{2}\pi$$

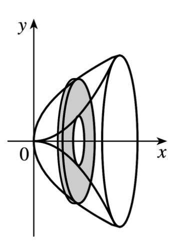




7. A cross–section is a washer (annulus) with inner radius x^2 and outer radius $\sqrt{x}$, so its area is $A(x)=\pi\left(\sqrt{x}\right)^2-\pi\left(x^2\right)^2=\pi\left(x-x^4\right)$.

$$V=\int_0^1 A(x)\,dx=\pi\int_0^1\left(x-x^4\right)dx=\pi\left[\frac{1}{2}x^2-\frac{1}{5}x^5\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{5}\right)=\frac{3\pi}{10}$$





19. R_1 about OA (the line $y=0$): $V=\int_0^1 A(x)\,dx=\int_0^1 \pi (x^3)^2\,dx=\pi\int_0^1 x^6\,dx=\pi\left[\frac{1}{7}x^7\right]_0^1=\frac{\pi}{7}$

21. R_1 about AB (the line $x=1$):

$$V=\int_0^1 A(y)\,dy=\int_0^1 \pi\left(1-\sqrt[3]{y}\right)^2 dy=\pi\int_0^1 (1-2y^{1/3}+y^{2/3})\,dy$$

$$=\pi\left[y-\frac{3}{2}y^{4/3}+\frac{3}{5}y^{5/3}\right]_0^1=\pi\left(1-\frac{3}{2}+\frac{3}{5}\right)=\frac{\pi}{10}$$

23. R_2 about OA (the line $y=0$):

$$V=\int_0^1 A(x)\,dx=\int_0^1\left[\pi(1)^2-\pi\left(\sqrt{x}\right)^2\right]dx=\pi\int_0^1 (1-x)\,dx=\pi\left[x-\frac{1}{2}x^2\right]_0^1=\pi\left(1-\frac{1}{2}\right)=\frac{\pi}{2}$$

25. R_2 about AB (the line $x=1$):

$$V=\int_0^1 A(y)\,dy=\int_0^1\left[\pi(1)^2-\pi(1-y^2)^2\right]dy=\pi\int_0^1\left[1-(1-2y^2+y^4)\right]dy$$

$$=\pi\int_0^1 (2y^2-y^4)\,dy=\pi\left[\frac{2}{3}y^3-\frac{1}{5}y^5\right]_0^1=\pi\left(\frac{2}{3}-\frac{1}{5}\right)=\frac{7\pi}{15}$$

27. R_3 about OA (the line $y=0$):

$$V=\int_0^1 A(x)\,dx=\int_0^1\left[\pi\left(\sqrt{x}\right)^2-\pi(x^3)^2\right]dx=\pi\int_0^1 (x-x^6)\,dx=\pi\left[\frac{1}{2}x^2-\frac{1}{7}x^7\right]_0^1=\pi\left(\frac{1}{2}-\frac{1}{7}\right)=\frac{5\pi}{14}\ .$$

Note: Let $= {}_1+$ mathpi5_2 $+R_3$. If we rotate about any of the segments OA , OC , AB , or BC , we obtain a right circular cylinder of height 1 and radius 1. Its volume is $\pi r^2 h=\pi(1)^2\cdot 1=\pi$. As a check for Exercises 19, 23, and 27, we can add the answers, and that sum must equal π. Thus,

$$\frac{\pi}{7}+\frac{\pi}{2}+\frac{5\pi}{14}=\left(\frac{2+7+5}{14}\right)\pi=\pi.$$

29. R_3 about AB (the line $x=1$):

$$V=\int_0^1 A(y)\,dy=\int_0^1\left[\pi(1-y^2)^2-\pi\left(1-\sqrt[3]{y}\right)^2\right]dy=\pi\int_0^1\left[(1-2y^2+y^4)-(1-2y^{1/3}+y^{2/3})\right]dy$$

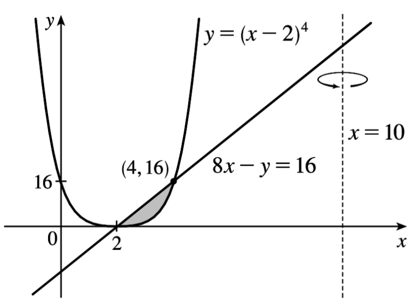
$$=\pi\int_0^1(-2y^2+y^4+2y^{1/3}-y^{2/3})\,dy=\pi\left[-\frac{2}{3}y^3+\frac{1}{5}y^5+\frac{3}{2}y^{4/3}-\frac{3}{5}y^{5/3}\right]_0^1$$

$$=\pi\left(-\frac{2}{3}+\frac{1}{5}+\frac{3}{2}-\frac{3}{5}\right)=\frac{13\pi}{30}$$

Note: See the note in Exercise 27. For Exercises 21, 25, and 29, we have
$\frac{\pi}{10}+\frac{7\pi}{15}+\frac{13\pi}{30}=\left(\frac{3+14+13}{30}\right)\pi=\pi$.

32. $y=(x-2)^4$ and $8x-y=16$ intersect when $(x-2)^4=8x-16=8(x-2)\Leftrightarrow(x-2)^4-8(x-2)=0\Leftrightarrow$
$(x-2)\left[(x-2)^3-8\right]=0\Leftrightarrow x-2=0$ or $x-2=2\Leftrightarrow x=2$ or 4. $y=(x-2)^4\Rightarrow x-2=\pm\sqrt[4]{y}\Rightarrow x=2+\sqrt[4]{y}$ [since $x\geq 2$].
$8x-y=16\Rightarrow 8x=y+16\Rightarrow x=\frac{1}{8}y+2$.

$$V=\pi\int_0^{16}\left\{\left[10-\left(\frac{1}{8}y+2\right)\right]^2-\left[10-\left(2+\sqrt[4]{y}\right)\right]^2\right\}dy$$



34. $V=\pi\int_0^{\pi}\left[(\sin x+2)^2-2^2\right]dx$

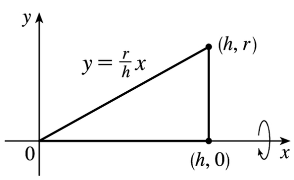
41. $\pi\int_0^{\pi/2}\cos^2 x\,dx$ describes the volume of the solid obtained by rotating the region
$R=\left\{(x,y)\,|\,0\leq x\leq\frac{\pi}{2},0\leq y\leq\cos x\right\}$ of the xy –plane about the x –axis.

43. $\pi\int_0^1 \left(y^4 - y^8\right) dy = \pi\int_0^1 \left[\left(y^2\right)^2 - \left(y^4\right)^2\right] dy$ describes the volume of the solid obtained by rotating the region $R = \left\{ (x,y) \mid 0 \le y \le 1, y^4 \le x \le y^2 \right\}$ of the xy –plane about the y –axis.

47. We'll form a right circular cone with height h and base radius r by revolving the line $y = \frac{r}{h} x$ about the x –axis.

$$V = \pi\int_0^h \left(\frac{r}{h} x\right)^2 dx = \pi\int_0^h \frac{r^2}{h^2} x^2 dx = \pi \frac{r^2}{h^2} \left[\frac{1}{3} x^3\right]_0^h$$

$$= \pi \frac{r^2}{h^2} \left(\frac{1}{3} h^3\right) = \frac{1}{3} \pi r^2 h$$



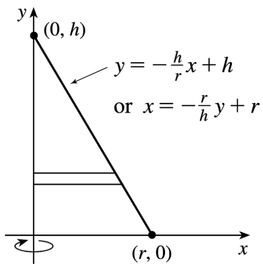
Another solution: Revolve $x = -\frac{r}{h} y + r$ about the y –axis.

$$V = \pi\int_0^h \left(-\frac{r}{h} y + r\right)^2 dy = \pi\int_0^h \left[\frac{r^2}{h^2} y^2 - \frac{2r^2}{h} y + r^2\right] dy$$

$$= \pi \left[\frac{r^2}{3h^2} y^3 - \frac{r^2}{h} y^2 + r^2 y\right]_0^h = \pi \left(\frac{1}{3} r^2 h - r^2 h + r^2 h\right) = \frac{1}{3} \pi r^2 h$$

* Or use substitution with $u = r - \frac{r}{h} y$ and $du = -\frac{r}{h} dy$ to get

$$\pi\int_r^0 u^2 \left(-\frac{h}{r}\right) du = -\pi \frac{h}{r} \left[\frac{1}{3} u^3\right]_r^0 = -\pi \frac{h}{r} \left(-\frac{1}{3} r^3\right) = \frac{1}{3} \pi r^2 h.$$


y
(0, h)
$y = -\frac{h}{r}x + h$
or $x = -\frac{r}{h}y + r$
(r, 0)
x