1. **(a)**

First Number	Second Number	Product
1	22	22
2	21	42
3	20	60
4	19	76
5	18	90
6	17	102
7	16	112
8	15	120
9	14	126
10	13	130
11	12	132

We needn't consider pairs where the first number is larger than the second, since we can just interchange the numbers in such cases. The answer appears to be 11 and 12 , but we have considered only integers in the table.

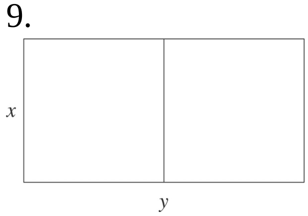
(b) Call the two numbers x and y. Then $x+y=23$, so $y=23-x$. Call the product P. Then $P=xy=x(23-x)=23x-x^2$, so we wish to maximize the function $P(x)=23x-x^2$. Since $P'(x)=23-2x$, we see that $P'(x)=0 \Leftrightarrow x=\frac{23}{2}=11.5$. Thus, the maximum value of P is $P(11.5)=(11.5)^2=132.25$ and it occurs when $x=y=11.5$.

Or: Note that $P''(x)=-2<0$ for all x , so P is everywhere concave downward and the local maximum at $x=11.5$ must be an absolute maximum.

3. The two numbers are x and $\frac{100}{x}$, where $x>0$. Minimize $f(x)=x+\frac{100}{x}$. $f'(x)=1-\frac{100}{x^2}=\frac{x^2-100}{x^2}$.

The critical number is $x=10$. Since $f'(x)<0$ for $0<x<10$ and $f'(x)>0$ for $x>10$, there is an absolute minimum at $x=10$. The numbers are 10 and 10.

9.

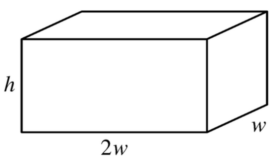


$xy=1.5\times 10^6$, so $y=1.5\times 10^6/x$. Minimize the amount of fencing, which is

$3x+2y=3x+2\left(1.5\times 10^6/x\right)=3x+3\times 10^6/x=F(x)$. $F'(x)=3-3\times 10^6/x^2=3\left(x^2-10^6\right)/x^2$. The critical number is $x=10^3$ and $F'(x)<0$ for $0<x<10^3$ and $F'(x)>0$ if $x>10^3$, so the absolute minimum occurs when $x=10^3$ and $y=1.5\times 10^3$. The field should be 1000 feet by 1500 feet with the middle fence parallel to the short side of the field.

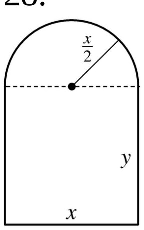
11. Let b be the length of the base of the box and h the height. The surface area is $1200=b^2+4hb\Rightarrow h=\left(1200-b^2\right)/(4b)$. The volume is $V=b^2h=b^2\left(1200-b^2\right)/4b=300b-b^3/4\Rightarrow V'(b)=300-\frac{3}{4}b^2$. $V'(b)=0\Rightarrow 300=\frac{3}{4}b^2\Rightarrow b^2=400\Rightarrow b=\sqrt{400}=20$. Since $V'(b)>0$ for $0<b<20$ and $V'(b)<0$ for $b>20$, there is an absolute maximum when $b=20$ by the First Derivative Test for Absolute Extreme Values (see page 280). If $b=20$, then $h=\left(1200-20^2\right)/(4\cdot 20)=10$, so the largest possible volume is $b^2h=(20)^2(10)=4000\text{ cm}^3$.

12.



$V=lwh\Rightarrow 10=(2w)(w)h=2w^2h$, so $h=5/w^2$. The cost is $10\left(2w^2\right)+6\left[2(2wh)+2(hw)\right]=20w^2+36wh$, so $C(w)=20w^2+36w\left(5/w^2\right)=20w^2+180/w$. $C'(w)=40w-180/w^2=40\left(w^3-\frac{9}{2}\right)\Big/w^2\Rightarrow w=\sqrt[3]{\frac{9}{2}}$ is the critical number. There is an absolute minimum for C when $w=\sqrt[3]{\frac{9}{2}}$ since $C'(w)<0$ for $0<w<\sqrt[3]{\frac{9}{2}}$ and $C'(w)>0$ for $w>\sqrt[3]{\frac{9}{2}}$. $C\left(\sqrt[3]{\frac{9}{2}}\right)=20\left(\sqrt[3]{\frac{9}{2}}\right)^2+\frac{180}{\sqrt[3]{9/2}}\approx\$163.54$.

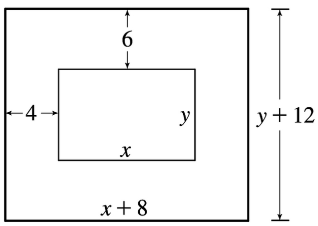
28.



Perimeter$=30\Rightarrow 2y+x+\pi\left(\frac{x}{2}\right)=30\Rightarrow y=\frac{1}{2}\left(30-x-\frac{\pi x}{2}\right)=15-\frac{x}{2}-\frac{\pi x}{4}$. The area is the area of the rectangle plus the area of the semicircle, or $xy+\frac{1}{2}\pi\left(\frac{x}{2}\right)^2$, so

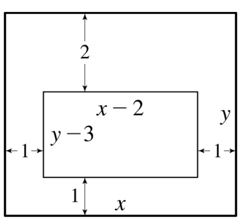
$A(x)=x\left(15-\dfrac{x}{2}-\dfrac{\pi x}{4}\right)+\dfrac{1}{8}\pi x^2=15x-\dfrac{1}{2}x^2-\dfrac{\pi}{8}x^2$. $A'(x)=15-\left(1+\dfrac{\pi}{4}\right)x=0\Rightarrow x=\dfrac{15}{1+\pi/4}=\dfrac{60}{4+\pi}$.
$A''(x)=-\left(1+\dfrac{\pi}{4}\right)<0$, so this gives a maximum. The dimensions are $x=\dfrac{60}{4+\pi}$ ft and
$y=15-\dfrac{30}{4+\pi}-\dfrac{15\pi}{4+\pi}=\dfrac{60+15\pi-30-15\pi}{4+\pi}=\dfrac{30}{4+\pi}$ ft, so the height of the rectangle is half the base.

29.



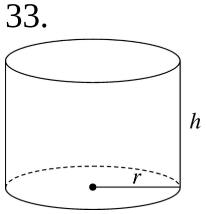
$xy=384\Rightarrow y=384/x$. Total area is $A(x)=(8+x)(12+384/x)=12(40+x+256/x)$, so
$A'(x)=12\left(1-256/x^2\right)=0\Rightarrow x=16$. There is an absolute minimum when $x=16$ since $A'(x)<0$ for $0<x<16$ and $A'(x)>0$ for $x>16$. When $x=16$, $y=384/16=24$, so the dimensions are 24 cm and 36 cm.

30.



$xy=180$, so $y=180/x$. The printed area is $(x-2)(y-3)=(x-2)(180/x-3)=186-3x-360/x=A(x)$.
$A'(x)=-3+360/x^2=0$ when $x^2=120\Rightarrow x=2\sqrt{30}$. This gives an absolute maximum since $A'(x)>0$ for $0<x<2\sqrt{30}$ and $A'(x)<0$ for $x>2\sqrt{30}$. When $x=2\sqrt{30}$, $y=180/(2\sqrt{30})$, so the dimensions are $2\sqrt{30}$ in. and $90/\sqrt{30}$ in.

33.

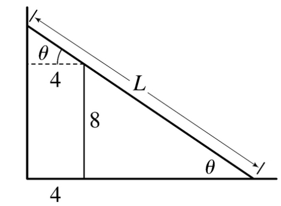


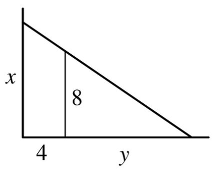
The volume is $V=\pi r^2 h$ and the surface area is $S(r)=\pi r^2+2\pi rh=\pi r^2+2\pi r\left(\dfrac{V}{\pi r^2}\right)=\pi r^2+\dfrac{2V}{r}$.

$S'(r)=2\pi r-\dfrac{2V}{r^2}=0\Rightarrow 2\pi r^3=2V\Rightarrow r=\sqrt[3]{\dfrac{V}{\pi}}$ cm.

This gives an absolute minimum since $S'(r)<0$ for $0<r<\sqrt[3]{\frac{V}{\pi}}$ and $S'(r)>0$ for $r>\sqrt[3]{\frac{V}{\pi}}$. When $r=\sqrt[3]{\frac{V}{\pi}}$, $h=\frac{V}{\pi r^2}=\frac{V}{\pi(V/\pi)^{2/3}}=\sqrt[3]{\frac{V}{\pi}}$ cm.

34.

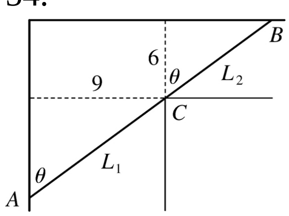




$L=8\theta+4\sec\theta$, $0<\theta<\frac{\pi}{2}$, $\frac{dL}{d\theta}=-8\theta\cot\theta+4\sec\theta\tan\theta=0$ when $\sec\theta\tan\theta=2\theta\cot\theta \Leftrightarrow \tan^3\theta=2 \Leftrightarrow$ $\tan\theta=\sqrt[3]{2} \Leftrightarrow \theta=\tan^{-1}\sqrt[3]{2}$.
$dL/d\theta<0$ when $0<\theta<\tan^{-1}\sqrt[3]{2}$, $dL/d\theta>0$ when $\tan^{-1}\sqrt[3]{2}<\theta<\frac{\pi}{2}$, so L has an absolute minimum when $\theta=\tan^{-1}\sqrt[3]{2}$, and the shortest ladder has length $L=8\frac{\sqrt{1+2^{2/3}}}{2^{1/3}}+4\sqrt{1+2^{2/3}}\approx 16.65$ ft.

Another method: Minimize $L^2=x^2+(4+y)^2$, where $\frac{x}{4+y}=\frac{8}{y}$.

54.



Paradoxically, we solve this maximum problem by solving a minimum problem. Let L be the length of the line ACB going from wall to wall touching the inner corner C. As $\theta\to 0$ or $\theta\to\frac{\pi}{2}$, we have $L\to\infty$ and there will be an angle that makes L a minimum. A pipe of this length will just fit around the corner.

From the diagram, $L = L_1 + L_2 = 9\csc\theta + 6\sec\theta \Rightarrow dL/d\theta = -9\csc\theta\ \cot\theta + 6\sec\theta\ \tan\theta = 0$ when $6\sec\theta\ \tan\theta = 9\csc\theta\ \cot\theta \Leftrightarrow \tan^3\theta = \frac{9}{6} = 1.5 \Leftrightarrow \tan\theta = \sqrt[3]{1.5}$. Then $\sec^2\theta = 1 + \left(\frac{3}{2}\right)^{2/3}$ and $\csc^2\theta = 1 + \left(\frac{3}{2}\right)^{-2/3}$, so the longest pipe has length

$$L = 9\left[1 + \left(\frac{3}{2}\right)^{-2/3}\right]^{1/2} + 6\left[1 + \left(\frac{3}{2}\right)^{2/3}\right]^{1/2} \approx 21.07 \text{ ft.}$$

Or, use $\theta = \tan^{-1}\left(\sqrt[3]{1.5}\right) \approx 0.852 \Rightarrow L = 9\ \theta + 6\sec\theta \approx 21.07$ ft.