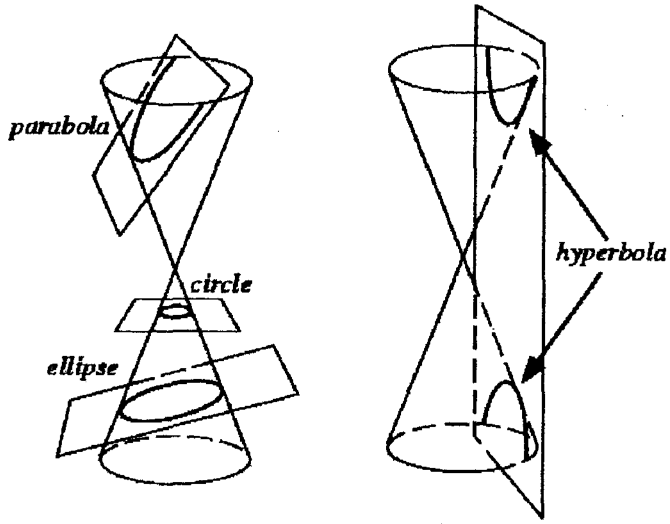
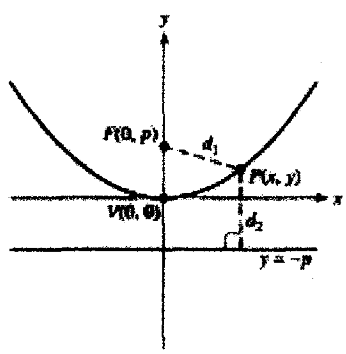
MAC 1140 Conic Sections 40 points

A conic section is the intersection of a plane and a cone. By changing the angle and location of intersection, we can produce an ellipse, parabola or hyperbola. Note: a circle is an example of an ellipse.



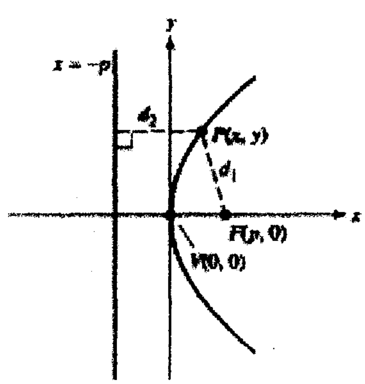
Parabolas

A parabola is the set of points in a plane equidistant from a fixed point and a fixed line. The fixed point is called the focus and the fixed line is called the directrix.



Vertical Axis

The parabola with a focus at $(0,p)$ and directrix $y = -p$ has equation $x^2 = 4py$. The parabola opens upward if $p > 0$ and downward if $p < 0$.

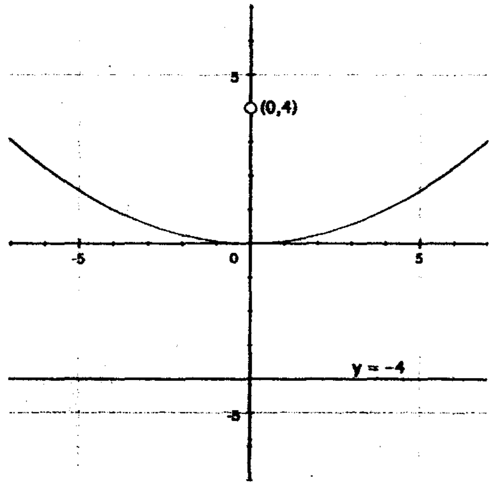


Horizontal Axis

The parabola with a focus at $(p,0)$ and directrix $x = -p$ has equation $y^2 = 4px$. The parabola opens to the right if $p > 0$ and to the left if $p < 0$.

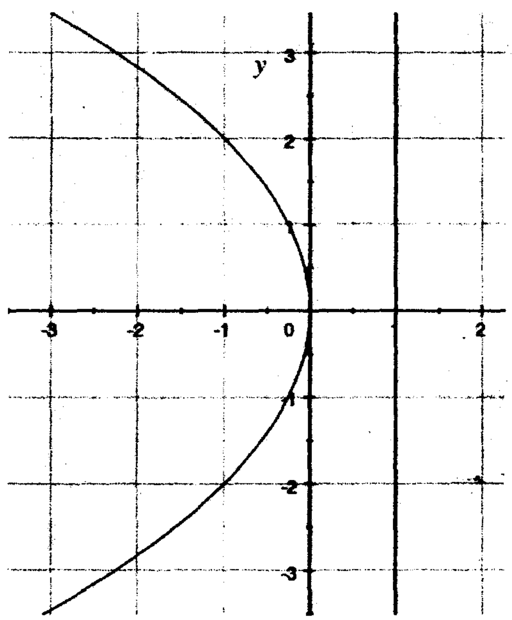
Example: Sketch a graph of the parabola $16y = x^2$. Label the vertex, focus and directrix.

Solution: The equation $16y = x^2$ is in the form $x^2 = 4py$, where $16 = 4p$. Therefore, the parabola has a vertical axis with $p = 4$. Since $p > 0$, the parabola opens upward. The focus is located at $(0,p)$ or $(0,4)$ and the directrix is $y = -p$ or $y = -4$.

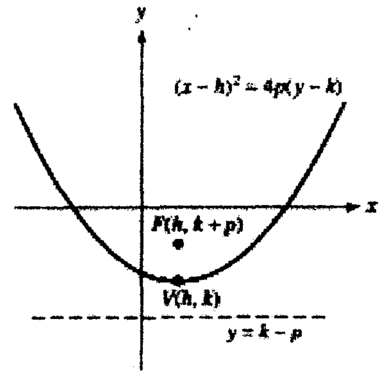


Example: Find the equation of a parabola with focus (-1,0) and vertex at (0,0).

Solution: A parabola always opens toward the focus and away from the directrix. In this case the parabola opens to the left. It follows that $p < 0$ in the equation $y^2 = 4px$. The distance between the focus (-1,0) and the vertex (0,0) is 1, so $p = -1 < 0$. The equation of the parabola is $y^2 = 4px = 4(-1)x$ or $y^2 = -4x$.



Translations of Parabolas

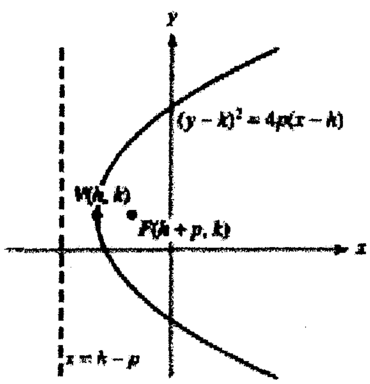


$(x - h)^2 = 4p(y - k)$

Vertical axis; vertex (h,k)

$p > 0$: opens upward; $p < 0$: opens downward

Focus: $(h,k + p)$; directrix: $y = k - p$



$(y - k)^2 = 4p(x - h)$

Horizontal axis; vertex (h,k)

$p > 0$: opens right; $p < 0$: opens left

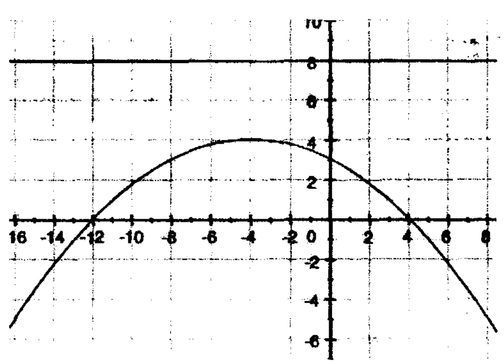
Focus: $(h + p,k)$; directrix: $x = h - p$

Example: Graph the parabola given by $y = -1/16(x + 4)^2 + 4$.

Solution: Rewrite the equation in the form $(x - h)^2 = 4p(y - k)$.

$$y = -1/16(x + 4)^2 + 4$$
$$y - 4 = -1/16(x + 4)^2$$
$$-16(y - 4) = (x + 4)^2$$
$$(x + 4)^2 = -16(y - 4)$$

It follows that the vertex is (-4,4), $4p = -16$ pr $p = -4$, and the parabola opens downward. The focus is located 4 units below the vertex, (-4,0) and the directrix is located 4 units above the vertex, $y = 8$.



Example: Find the equation of the parabola with focus (2,1) and directrix $x = -1$.

Solution: Since the directrix is to the left of the focus, the parabola opens to the right, $p > 0$ and satisfies the equation $(y - k)^2 = 4p(x - h)$. The vertex is located halfway between the directrix and the focus, so its coordinates are (0.5,1). The distance between the focus (2,1) and the vertex (0.5,1) is 1.5 so $p = 1.5$. The equation of the parabola is

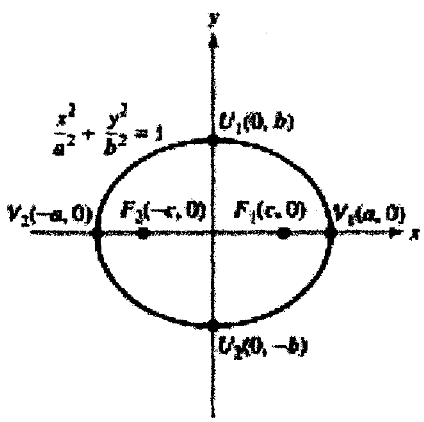
$$(y - 1)^2 = 6(x - 0.5)$$

Exercises

1. Graph the parabola. Label the vertex, focus, and directrix: $y^2 = 2x$ (3 points)
2. Find an equation of a parabola that satisfies the given conditions:
   Focus (2,3); Directrix $y = 4$ (4 points)

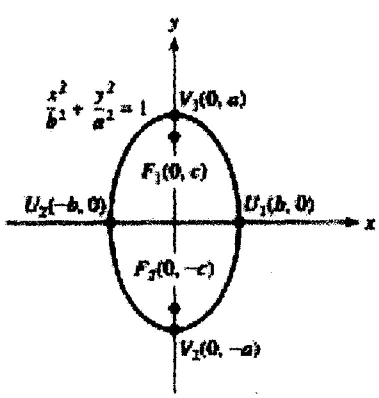
Ellipses

An ellipse is the set of points in a plane, the sum of whose distances from two fixed points is constant. Each fixed point is called a focus (plural foci) of the ellipse.



Horizontal Major Axis

The ellipse with center at the origin, horizontal major axis, and equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b > 0$ has vertices $(\pm a, 0)$, endpoints of the minor axis $(0, \pm b)$, and foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$ and $c \geq 0$.



Vertical Major Axis

The ellipse with center at the origin, vertical major axis, and equation $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1,\ a > b > 0$ has vertices $(0, \pm a)$, endpoints of the minor axis $(\pm b, 0)$, and foci $(0, \pm c)$, where $c^2 = a^2 - b^2$ and $c \geq 0$.

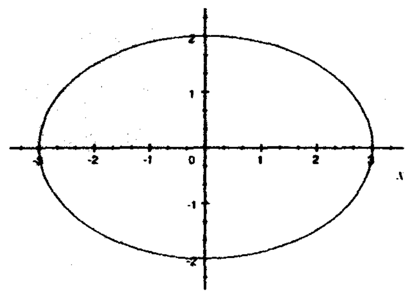
Note: If $a = b$, the ellipse is a circle with radius $r = a$ and center (0,0).

Example: Sketch the graph of the ellipse $4x^2 + 9y^2 = 36$.

Solution: The equation $4x^2 + 9y^2 = 36$ can be put into standard form by dividing both sides by 36.

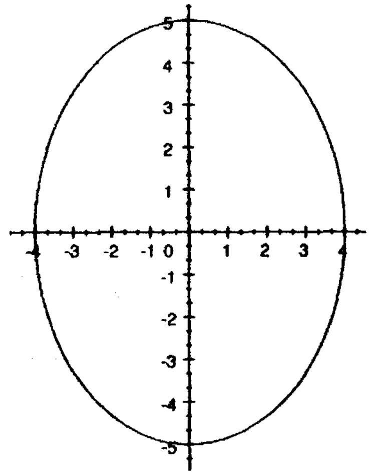
$$4x^2 + 9y^2 = 36 \quad \Rightarrow \quad \frac{4x^2}{36} + \frac{9y^2}{36} = \frac{36}{36} \quad \Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{4} = 1$$

The ellipse has a horizontal major axis with $a = 3$ and $b = 2$. The value of c is found using $c^2 = a^2 - b^2 = 9 - 4 = 5$ or $c = \sqrt{5}$. The ellipse has foci $(\pm\sqrt{5}, 0)$, vertices $(\pm 3, 0)$, and endpoints of the minor axis located at $(0, \pm 2)$.



Example: Find the equation of an ellipse, centered at the origin with foci $(0, \pm 3)$ and vertices $(0, \pm 5)$.

Solution: Since the vertices line on the y-axis, the ellipse has a vertical major axis. Its standard equation has the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$. Since the foci are $(0, \pm 3)$, $c = 3$ and since the vertices are $(0, \pm 5)$, $a = 5$. The value of b can be found by using $c^2 = a^2 - b^2$. Thus, $b^2 = a^2 - c^2 = 25 - 9 = 16$ so $b = 4$. The equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{25} = 1$.



Translations of Ellipses

An ellipse with center (h,k), and either a horizontal or vertical major axis, satisfies one of the following equations, where $a > b > 0$ and $c^2 = a^2 - b^2$.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Major Axis: horizontal; foci: $(h \pm c, k)$

Vertices: $(h \pm a, k)$

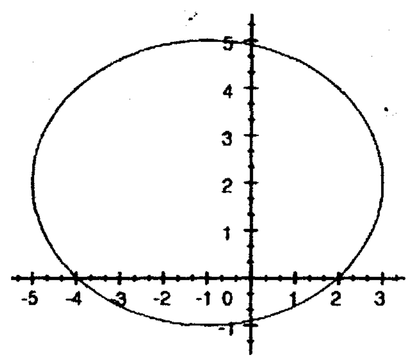
$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

Major Axis: vertical; foci: $(h, k \pm c)$

Vertices: $(h, k \pm a)$

Example: Sketch the graph of the ellipse $\frac{(x+1)^2}{16} + \frac{(y-2)^2}{9} = 1$. Identify the foci and vertices.

Solution: The ellipse has a horizontal major axis and its center is (-1,2). Since $a^2 = 16$ and $b^2 = 9$, we find $c^2 = 16 - 9 = 7$. Therefore, $a = 4$, $b = 3$, and $c = \sqrt{7}$. The vertices are located 4 units to the right and left of (-1,2). They are (-5,2) and (3,2). The foci are located $\sqrt{7}$ units to the right and left of (-1,2). They are $(-1 \pm \sqrt{7}, 2)$. The endpoints of the minor axis are located 3 units above and below (-1,2). They are (-1,-1) and (-1,5).



Example: Find an equation of an ellipse with center at (0,1), vertex at (0,4), and foci at $(0, 1+\sqrt{5})$.

Solution: The ellipse is centered at (0,1) and has a vertical major axis. Its standard equation has the form $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$. The distance from the center to the vertex is 3, so $a = 3$. The distance from the center to the focus is $\sqrt{5}$ so $c = \sqrt{5}$. We can find b by using $c^2 = a^2 - b^2$. Thus $b^2 = a^2 - c^2 = 9 - 5 = 4$ so $b = 2$. The standard equation is $\frac{x^2}{4} + \frac{(y-1)^2}{9} = 1$.

Exercises

Sketch the graph of the ellipse. Label the foci and the endpoints of each axis.

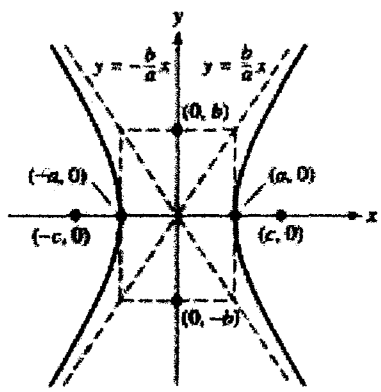
3. $\frac{x^2}{16}+\frac{y^2}{36}=1$ (4 points)

4. $\frac{(x-4)^2}{9}+\frac{(y+2)^2}{25}=1$ (5 points)

5. Find an equation of an ellipse that satisfies the given conditions:
Center (0,0); Foci (±3,0); Vertices (±5,0) (4 points)

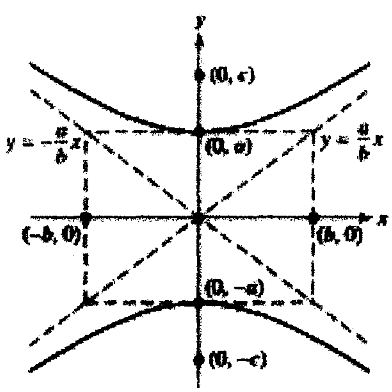
Hyperbolas

A hyperbola is the set of points in a plane, the difference of whose distances from two fixed points is constant. Each fixed point is called a focus of the hyperbola.



Horizontal transverse axis

The hyperbola with center at the origin and equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ has asymptotes $y=\pm\frac{b}{a}x$, vertices $(\pm a,0)$, and foci $(\pm c,0)$, where $c^2=a^2+b^2$.



Vertical transverse axis

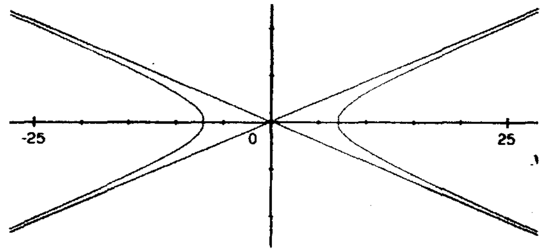
The hyperbola with center at the origin and equation $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ has asymptotes $y=\pm\frac{a}{b}x$, vertices $(0,\pm a)$, and foci $(0,\pm c)$, where $c^2=a^2+b^2$.

Example: Sketch a graph of $\frac{x^2}{49}-\frac{y^2}{9}=1$, including the asymptotes. Give the coordinates of the foci.

Solution: The equation is in standard form with $a=7$ and $b=3$. It has a horizontal transverse axis with vertices $(\pm 7,0)$. The endpoints of the conjugate axis are $(0,\pm 3)$. For the foci we need to find c using

$c^2 = a^2 + b^2 = 49 + 9 = 58$ so $c = \sqrt{58}$. The foci are $(\pm\sqrt{58}, 0)$. The asymptotes are $y = \pm\frac{b}{a}x$ or

$y = \pm\frac{3}{7}x$.



Example: Determine the equation of the hyperbola, centered at the origin, with foci $(0, \pm 13)$ and vertices $(0, \pm 5)$.

Solution: Since the hyperbola is centered at the origin with vertical transverse axis, its equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. The distance from the vertices to the center is 5 so $a = 5$. The distance from the foci to the center is 13, so $c = 13$. We can find b by using $c^2 = a^2 + b^2$. Thus $b^2 = c^2 - a^2 = 169 - 24 = 144$, so $b = 12$. The equation of the hyperbola is $\frac{y^2}{25} - \frac{x^2}{144} = 1$.

Translations of Hyperbolas

A hyperbola with center (h, k), and either a horizontal or vertical transverse axis, satisfies one of the following equations, where $c^2 = a^2 + b^2$.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Horizontal transverse axis

Vertices: $(h \pm a, k)$; foci: $(h \pm c, k)$

Asymptotes: $y = \pm\frac{b}{a}(x-h) + k$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

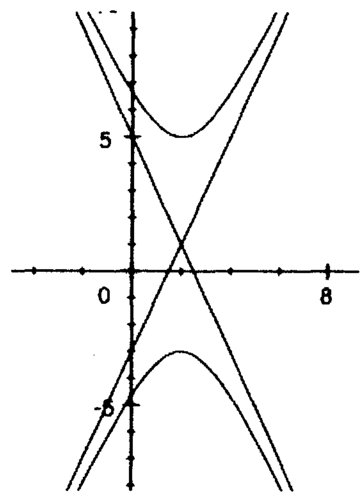
Vertical transverse axis

Vertices: $(h, k \pm a)$; foci: $(h, k \pm c)$

Asymptotes: $y = \pm\frac{a}{b}(x-h) + k$

Example: Sketch the graph of the hyperbola $\frac{(y-1)^2}{16} - \frac{(x-2)^2}{4} = 1$.

Solution: The hyperbola has a vertical transverse axis and its center is (2,1). Since $a^2 = 16$ and $b^2 = 4$, we find $c^2 = a^2 + b^2 = 16 + 4 = 20$. Thus $a = 4$, $b = 2$, and $c = 2\sqrt{5}$. The vertices are located 4 units above and below the center of the hyperbola at (2,5) and (2,-3). The foci are located $2\sqrt{5}$ units above and below the center of the hyperbola at $(2, 1 \pm 2\sqrt{5})$. The asymptotes are given by $y = \pm\frac{a}{b}(x-h)+k$ or $y = \pm 2(x-2)+1$.



Example: Find the standard equation of a hyperbola with center (-2,2), focus (-2,4), and vertex (-2,3).

Solution: The equation of a hyperbola with a vertical transverse axis and center at (-2,2) is given by $\frac{(y-2)^2}{a^2} - \frac{(x+2)^2}{b^2} = 1$. The distance from the center to the vertex is 1 so $a = 1$. The distance from the center to the focus is 2 so $b = 2$. The equation of the hyperbola is $(y-2)^2 - \frac{(x+2)^2}{4} = 1$

Exercises

Sketch a graph of the hyperbola, including the asymptotes. Give the coordinates of the foci and the equations of the asymptotes.

6. $9y^2 - 25x^2 = 225$ (5 points)

7. $\frac{(x+1)^2}{16} - \frac{(y+3)^2}{9} = 1$ (5 points)

8. Find the standard equation of a hyperbola with center (h,k) that satisfies the given conditions:
Center (0,0); foci $(0, \pm\sqrt{20})$; vertices $(0, \pm 4)$ (4 points)

9. Suppose the foci for a certain conic section are (4,2) and (-4,2) and the vertices are located at (6,2) and (-6,2).

A) Is the conic section a parabola, an ellipse, or a hyperbola? Explain how you know. (3 points)

B) Find the equation of the conic section described. (3 points)