OBJECTIVE 8

Limits

The Definition of a Limit

For a fixed real number L and constant c,

$$\lim_{x \to c} f(x) = L$$

means that values of $f(x)$ can be made arbitrarily close (infinitely close) to L, by taking values of x sufficiently close to c, $x \neq c$.

Right-Hand Limits

For a fixed real number L and constant c,

$$\lim_{x \to c^+} f(x) = L$$

means that values of $f(x)$ can be made arbitrarily close (infinitely close) to L, by taking values of x sufficiently close to c and to the right of $c (x > c)$.

Left-Hand Limits

For a fixed real number L and constant c,

$$\lim_{x \to c^-} f(x) = L$$

means that values of $f(x)$ can be made arbitrarily close (infinitely close) to L, by taking values of x sufficiently close to c and to the left of $c (x < c)$.

USING A TABLE TO EVALUATE A LIMIT

Use a table of values to evaluate each function as x approaches the value indicated.

1. $p(x) = \frac{x^2-3x-10}{2x+4}; x \to -2$
2. $g(x) = \frac{x}{\ln x}; x \to 0$
3. $f(x) = \frac{x^2-10x+24}{\sqrt{x^2+6x+9}}; x \to 6$

Properties of Limits

Given that the limits $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist,

(I) $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

the limit of a sum is the sum of the limits

(II) $\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$

the limit of a difference is the difference of the limits

(III) $\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$

the limit of a product is the product of the limits

(IV) $\lim_{x\to c}[kg(x)] = k\lim_{x\to c} g(x)$, k a constant

the limit of a constant times a function is the constant times the limit of the function

(V) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, *provided* $\lim_{x\to c} g(x) \neq 0$

the limit of a quotient is the quotient of the limits

By repeatedly applying property III with $f(x) = g(x)$, we obtain the property related to the limit of a power.

(VI) $\lim_{x \to c} [f(x)]^n = [\lim_{x \to c} f(x)]^n$, for n a natural number

the limit of a power is the power of the limit

A similar property holds for the limit of an nth root.

(VII) $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$, for $n \in \mathbb{N}$, $n > 1$ [if n is even, then $f(x) > 0$]

the limit of an nth root is the nth root of the limit

Basic Limits	
1. $\lim_{x \to c} k = k$	**2.** $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$, n a natural number	**4.** $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, n a natural number, $n > 1$ (if n is even, then $c > 0$)

EVALUATING A LIMIT ALGEBRAICALLY

Evaluate each limit.

1. $\lim_{x \to 2} (x^4 - 2x + 5)$
2. $\lim_{x \to 2} \frac{x^2 + 3x - 10}{x - 2}$
3. $\lim_{x \to -5} (2x + \sqrt{4 - x})$
4. $\lim_{x \to 3} \frac{2x^2 - 3x - 9}{x - 3}$

EVALUATING A LIMIT ALGEBRAICALLY

Evaluate each limit.

1. $\lim_{x\to 16}\frac{x-16}{4-\sqrt{x}}$
2. $\lim_{h\to 0}\frac{\frac{2}{x+h-1}-\frac{2}{x-1}}{h}$
3. $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$
4. $\lim_{h\to 0}\frac{\frac{3}{x+h+2}-\frac{3}{x+2}}{h}$

PIECEWISE-DEFINED FUNCTIONS

Find the specified limit for the function given.

1. $\lim_{x\to 10} g(x);\ g(x)=\begin{cases}-(x+1), & x\le 10\\ \log x, & x>10\end{cases}$
2. $\lim_{x\to 5} f(x);\ f(x)=\begin{cases}2x^2-7, & x<5\\ 3-2x, & x\ge 5\end{cases}$

INFINITE LIMITS

Find the limit, if it exists.

1. $\lim_{x\to 1^-}\frac{16x^2}{(x-1)^2}$
2. $\lim_{x\to 1^+}\frac{16x^2}{(x-1)^2}$
3. $\lim_{x\to 1}\frac{16x^2}{(x-1)^2}$

INFINITE LIMITS

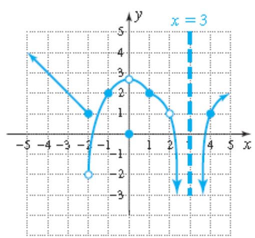
Find the limit, if it exists.

1. $\lim\limits_{x \to 3^-} \dfrac{4}{x-3}$

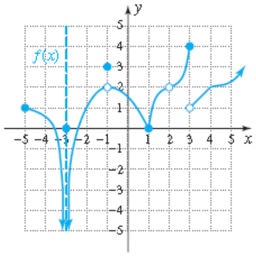
2. $\lim\limits_{x \to 3^+} \dfrac{4}{x-3}$

3. $\lim\limits_{x \to 3} \dfrac{4}{x-3}$

GRAPHICAL LIMITS



GRAPHICAL LIMITS



Limits at Positive Infinity

For a function f defined on an interval (c, ∞),

$$\lim_{x \to \infty} f(x) = L$$

means that values of $f(x)$ can be made arbitrarily close to L, by taking values of x sufficiently large and positive.

Limits at Negative Infinity

For a function f defined on an interval $(-\infty, c)$

$$\lim_{x \to -\infty} f(x) = L$$

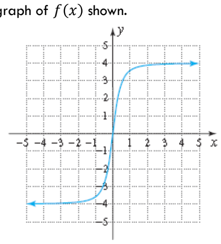
means that values of $f(x)$ can be made arbitrarily close to L, by taking values of x sufficiently large and negative.

LIMITS AT INFINITY

Find the following limits for the graph of $f(x)$ shown.

$\lim_{x \to \infty} f(x)$

$\lim_{x \to -\infty} f(x)$



Horizontal Asymptotes

The line $y = L$ is a horizontal asymptote if

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to \infty} f(x) = L$$

Limits of Reciprocal Powers

For any positive integer k,

$$\lim_{x \to -\infty} \frac{1}{x^k} = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{1}{x^k} = 0$$

LIMITS OF RATIONAL FUNCTIONS AT INFINITY

Evaluate the limit:

1. $\lim_{x\to\infty} \frac{2x^2 - 1}{3x^2 + 2}$
2. $\lim_{x\to\infty} \frac{5x^2 + 3x}{8 - 7x^2}$
3. $\lim_{x\to\infty} \frac{x - 4}{x^2 + 9}$