OBJECTIVE 6

Sequences and Series

SEQUENCE

A sequence is an ordered list of numbers, called terms.

Examples:

$$a_1, a_2, a_3, \ldots$$

$$1, 2, 3, 4, 5, \ldots$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

$$1, -1, 1, -1, 1, \ldots$$

$$3, 1, 4, 1, 5, 9, 2, 6, 5, 4, \ldots$$

SEQUENCE

SEQUENCE

An **infinite sequence** is a function that has the set of natural numbers as its domain. A **finite sequence** is a function with domain $D = \{1, 2, 3, \ldots, n\}$, for some fixed natural number n.

Example:

infinite sequence: $1, 2, 3, 4, 5, \ldots$

finite sequence: $1, 2, 3, 4, 5, \ldots, 10$

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.888

GENERAL TERM

When the sequence has a definite pattern, e can use a general term to describe the sequence:

$$1, 2, 3, 4, 5, \ldots, n, \ldots$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots$$

$$1, -1, 1, -1, 1, \ldots, (-1)^{n+1}, \ldots$$

GENERAL TERM

Find the general term of the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$$

NOTATION

We can use the general term to represent the sequence.

Example: $a_n = \frac{1}{n}$ is the general term of the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

FINDING TERMS OF A SEQUENCE

1. Find the first five terms of the recursively defined sequence

$$a_n = \frac{a_{n-1}}{2}, \quad a_1 = -8$$

2. Find the first five terms of the sequence

$$a_n = \frac{2^n}{n^2+3}$$

FACTORIAL

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

Calculate:

1. 3!
2. 5!
3. $\frac{7!}{5!}$
4. $\frac{3!9!}{4!2!}$

ARITHMETIC SEQUENCE

INFINITE ARITHMETIC SEQUENCE

An **infinite arithmetic sequence** is a linear function whose domain is the set of natural numbers.

***n*th TERM OF AN ARITHMETIC SEQUENCE**

In an arithmetic sequence with first term a_1 and common difference d, the nth term, a_n, is given by

$$a_n = a_1 + (n-1)d.$$

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.893-5

ARITHMETIC SEQUENCES

1. Is 2, 4, 6, 8, … arithmetic?
2. 5.1, 5.5, 5.9, 6.3, 6.7, … is an arithmetic sequence. Write out the next three terms and find the general term.
3. Find the 24th term of the sequence 0.1, 0.4, 0.7, 1, …

PARTIAL SUM OF ARITHMETIC SEQUENCE

The *n*th Partial Sum of an Arithmetic Sequence

Given an arithmetic sequence with first term a_1, the nth partial sum is given by

$$S_n = n\left(\frac{a_1 + a_n}{2}\right).$$

In words: The sum of an arithmetic sequence is the number of terms times the average of the first and last term.

$1 + 2 + 3 + 4 + 5 + \quad + 99 + 100$

2 + 99 = 101

1 + 100 = 101

PARTIAL SUM OF ARITHMETIC SEQUENCE

1. Find the sum of the first 75 positive, odd integers:
$$\sum_{k=1}^{75} (2k-1)$$
2. Find the sum:
$$\sum_{n=1}^{29} (4n-1)$$

GEOMETRIC SEQUENCE

INFINITE GEOMETRIC SEQUENCE

An **infinite geometric sequence** is a function defined by $f(n) = cr^{n-1}$, where c and r are nonzero constants. The domain of f is the set of natural numbers.

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.896

EXAMPLES

1. Determine the common ratio, the fifth term, and the n[th] term of the geometric sequence

$$7, \frac{14}{3}, \frac{28}{9}, \frac{56}{27}, \ldots$$

2. Classify the sequence 5, 2, -2, -6, -11 as arithmetic, geometric, or neither.
3. Find the 10[th] term of the sequence $3, -6, 12, -24, \ldots$

PARTIAL SUM OF GEOMETRIC SEQUENCE

The *n*th Partial Sum of a Geometric Sequence

Given a geometric sequence with first term a_1 and common ratio r, the nth partial sum (the sum of the first n terms) is

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} = \frac{a_1(1 - r^n)}{1 - r}, \; r \neq 1$$

In words: The sum of a geometric sequence is the difference of the first and $(n + 1)$st term, divided by 1 minus the common ratio.

PARTIAL SUM OF GEOMETRIC SEQUENCE

Let $s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$.

Then $rs = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n$

Then $s - rs = a - ar^n$

Then $s(1 - r) = a(1 - r^n)$, so $s = a\dfrac{1 - r^n}{1 - r}$ (if $r \neq 1$).

PARTIAL SUM OF A GEOMETRIC SEQUENCE

1. Find the sum:

$$\sum_{i=1}^{9} 3^i$$

2. Find the sum:

$$\sum_{j=1}^{7} 3\left(\frac{1}{5}\right)^{j-1}$$

3. If $a_2 = -5$ and $a_5 = \frac{1}{25}$, find S_5.

GEOMETRIC SERIES

Infinite Geometric Series

Given a geometric sequence with first term a_1 and $|r| < 1$, the sum of the related infinite series is given by

$$S_\infty = \frac{a_1}{1 - r}; r \neq 1$$

If $|r| > 1$, no finite sum exists.

GEOMETRIC SERIES

Determine whether the geometric series has a finite sum. If so, find it.

1. $3 + 6 + 12 + 24 + \cdots$
2. $9 + 3 + 1 + \cdots$
3. $4 + 8 + 16 + 32 + \cdots$
4. $-49 + (-7) + \left(-\frac{1}{7}\right) + \cdots$
5. $\sum_{1}^{\infty} \frac{3}{4}\left(\frac{2}{3}\right)^k$
6. $\sum_{k=1}^{\infty} 12\left(\frac{4}{3}\right)^k$