OBJECTIVE 5

Systems & Matrices

SOLVING SYSTEMS OF TWO EQUATIONS BY SUBSTITUTION:

1. Solve one of the equations for one of the variables in terms of the other.
2. Substitute this expression into the 2nd equation, which will yield an equation in one variable.
3. Solve the new equation.
4. Use the result of step 1 to find the other variable.

SOLVING SYSTEMS

Solve the system of linear equations:

1. $\begin{cases} 7x - 2y = 5 \\ x + 9y = 10 \end{cases}$

2. $\begin{cases} \frac{1}{2}x - \frac{3}{4}y = \frac{1}{2} \\ \frac{1}{5}x - \frac{3}{10}y = \frac{1}{5} \end{cases}$

3. $\begin{cases} 0.6x - 0.2y = 2 \\ -1.2x + 0.4y = 3 \end{cases}$

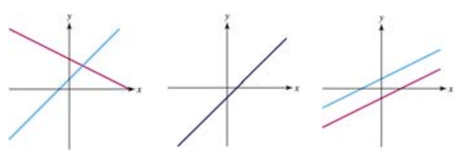
SOLVING SYSTEMS

Solve the system of linear equations:

1. $\begin{cases} 4x - 5y = 7 \\ 2x - 5 = y \end{cases}$
2. $\begin{cases} 2x - y = 6 \\ \frac{3}{4}x - 1 = y \end{cases}$

Not every system of equations has a unique solution. There are three possibilities:

1. The graphs may be the same line. We say this is a dependent system.
2. The graphs may be parallel but distinct lines. We say this is an inconsistent system.
3. The graphs may intersect in one and only one point. The system is said to be consistent and independent.



From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.755.

SOLVING A SYSTEM OF TWO EQUATIONS BY ELIMINATION

1. Choose one of the variables to eliminate. Multiply each equation by a suitable factor so that the coefficients of that variable are opposite.
2. Add the two new equations termwise.
3. Solve the resulting equation for the remaining variable.
4. Substitute the value found in step 3 into either of the original equations and solve for the other variable.

This is also called the method of linear combinations.

SOLVING A SYSTEM BY ELIMINATION

Solve the system of linear equations:

1. $\begin{cases} 4x+3y=8 \\ -2x+6y=1 \end{cases}$

2. $\begin{cases} -\frac{1}{3}x+\frac{1}{6}y=-1 \\ 2x-y=6 \end{cases}$

3. $\begin{cases} 5x-2y=7 \\ 10x-4y=6 \end{cases}$

SOLVING A SYSTEM BY ELIMINATION

Solve the system of linear equations:

1. $\begin{cases} 5x+9y=-22 \\ -10x+6y=-28 \end{cases}$

2. $\begin{cases} -\frac{2}{3}x+9y=47 \\ 3x-\frac{1}{5}y=-10 \end{cases}$

3. $\begin{cases} 4x-y=3 \\ 8x-2y=6 \end{cases}$

SOLVING A SYSTEM

A jeweler is commissioned to create a piece of artwork that will weigh 14 oz and consist of 75% gold. She has on hand two alloys that are 60% and 80% gold, respectively. How much of each should she use?

SOLVING A SYSTEM

An airplane flying due south from St. Louis, Missouri, to Baton Rouge, Louisiana, uses a strong, steady tailwind to complete the trip in only 2.5 hr. On the return trip, the same wind slows the flight and it takes 3 hr to get back. If the flight distance between these cities is 912 km, what is the cruising speed of the airplane (speed with no wind)? How fast is the wind blowing?

SOLVING A SYSTEM OF 3 EQUATIONS & 3 UNKNOWNS

1. Clear each equation of fractions and put it in standard form.
2. Choose two of the equations & eliminate one of the variables by forming a linear combination.
3. Choose a different pair of equations & eliminate the same variable.
4. Form a 2 x 2 system with the equations found in steps 2 & 3. Eliminate one of the variables from this 2 x 2 system using a linear combination.
5. Use back-substitution to solve.

SOLVING A SYSTEM

Solve the system:

1. $\begin{cases} -x-5y+2z=2 \\ x+y+2z=2 \\ 3x+y-4z=-10 \end{cases}$

2. $\begin{cases} x-2y+z=1 \\ x+y+2z=2 \\ 2x+3y+z=6 \end{cases}$

3. $\begin{cases} 2x-y+2z=6 \\ -x+y+z=0 \\ -x-3z=-6 \end{cases}$

4. $\begin{cases} 2x+y+3z=4 \\ -3x-y-4z=5 \\ x+y+2z=0 \end{cases}$

SOLVING A SYSTEM

Solve the system:

1. $\begin{cases} 2x+y-2z=-7 \\ x+y+z=-1 \\ -2y-z=-3 \end{cases}$

SOLVING A SYSTEM

A small business borrowed $225,000 from three different lenders to expand their product line. The interest rates were 5%, 6%, and 7%. Find how much was borrowed at each rate if the annual interest came to $13,000 and twice as much was borrowed at the 5% rate than was borrowed at the 7% rate.

PARTIAL FRACTION DECOMPOSITION

Decompose the expression into partial fractions:

1. $\frac{4x+11}{x^2+7x+10}$
2. $\frac{2}{(x+5)(x^2+7x+10)}$
3. $\frac{3x+11}{x^3-3x^2+x-3}$

PARTIAL FRACTION DECOMPOSITION

Write out the partial fraction template (DO NOT SOLVE!):

1. $\frac{5x-7}{3x^2+11x-4}$
2. $\frac{2x-1}{(x+3)^3(x^2+8)}$
3. $\frac{5x}{x^3-x^2+9x}$

PROCEDURE FOR SOLVING INEQUALITIES IN TWO VARIABLES:

1. If possible, solve for *y*. Graph the curve described by the equation.
2. If the form is $y \leq mx + b$ shade below the line
3. If the form is $y \geq mx + b$ shade above the line
4. Otherwise, use a test point to determine where to shade.
5. Use a dotted line for a strict inequality!

SOLVING INEQUALITIES

Graph the solution set to the inequality:

1. $y > 2x$
2. $2x + 3y \leq 6$

SOLVING A SYSTEM OF INEQUALITIES

Solve the system:

1. $\begin{cases} 2x + y \geq 4 \\ x - y < 2 \end{cases}$
2. $\begin{cases} 2x + y > 5 \\ x - 2y < -7 \end{cases}$

SOLVING A SYSTEM OF INEQUALITIES

As part of their retirement planning, James and Lily decide to invest up to $30,000 in two separate investment vehicles. The first is a bond issue paying 9% and the second is a money market certificate paying 5%. A financial adviser suggests they invest at least $10,000 in the certificate and not more than $15,000 in bonds. What various amounts can be invested in each?

MATRICES

A matrix is a rectangular array of elements:

$$\begin{bmatrix} 1 & -2 & 4 \\ 5 & 1 & 3 \\ -1 & 4 & 7 \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A matrix that has dimensions $m \times n$ has m rows and n columns.

A square matrix has the same number of rows and columns ($n \times n$).

EXAMPLE

Represent the system by an augmented matrix and state the dimensions of the matrix:

1. $\begin{cases} 3x-5y=4 \\ x+y=2 \end{cases}$

2. $\begin{cases} 2x+y+3z=4 \\ -3x-y-4z=5 \\ x+y+2z=0 \end{cases}$

ROW-ECHELON FORM

A matrix is in row-echelon form if it has ones down the main diagonal and zeros beneath:

$$\begin{bmatrix} 1 & 2 & -5 & 7 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

A matrix is in reduced row-echelon form if it has ones down the main diagonal and zeros above and below:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

EXAMPLES

Write the equations that correspond to the augmented matrix, then solve the system:

1. $\begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -1 & 2 & 8 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & -4 & 3/4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

GAUSSIAN ELIMINATION

MATRIX ROW TRANSFORMATIONS

For any augmented matrix representing a system of linear equations, the following row transformations result in an equivalent system of linear equations.

1. Any two rows may be interchanged.
2. The elements of any row may be multiplied by a nonzero constant.
3. Any row may be changed by adding to (or subtracting from) its elements a multiple of the corresponding elements of another row.

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.783

SOLVING A SYSTEM WITH A MATRIX

Solve the system:

1. $\begin{cases} x+3y-2z=3 \\ -x-2y+z=-2 \\ 2x-7y+z=1 \end{cases}$

2. $\begin{cases} 4x-y-z=0 \\ 4x-2y=0 \\ 2x+z=1 \end{cases}$

3. $\begin{cases} 2x-y-z=0 \\ x-y-z=-2 \\ 3x-2y-2z=-2 \end{cases}$

4. $\begin{cases} 2x-4y-z=2 \\ x+y-3z=10 \\ -x-7y+8z=2 \end{cases}$

SOLVING A SYSTEM WITH A MATRIX

Use technology to find the solution. Approximate values to the nearest thousandth.

$$\begin{cases} 12x-4y-7z=8 \\ -8x-6y+9z=7 \\ 34x+6y-2z=5 \end{cases}$$

OPERATIONS ON MATRICES

Matrix Addition

The sum of two $m \times n$ matrices A and B is the $m \times n$ matrix $A + B$, in which each element is the sum of the corresponding elements of A and B. This is written as $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$. If A and B have different dimensions, then $A + B$ is undefined.

Matrix Subtraction

The difference of two $m \times n$ matrices A and B is the $m \times n$ matrix $A - B$, in which each element is the difference of the corresponding elements of A and B. This is written as $A - B = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}]$. If A and B have different dimensions, then $A - B$ is undefined.

Multiplication of a Matrix by a Scalar

The product of a scalar (real number) k and an $m \times n$ matrix A is the $m \times n$ matrix kA, in which each element is k times the corresponding element of A. This is written as $kA = k[a_{ij}] = [ka_{ij}]$.

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.801

ADDING MATRICES

For the given matrices *A* and *B*, find each of the following:

$$A = \begin{bmatrix} 2 & -4 \\ -1 & 1/2 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 0 \\ 3 & 1/2 \\ -1 & 1 \end{bmatrix}$$

1. *A+B*
2. *B+A*
3. *A-B*

EXAMPLE

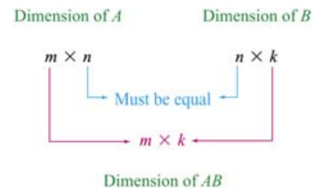
For the given matrices *A* and *B*, find each of the following:

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 3 & -4 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 & -5 \\ -3 & 1 & 2 \end{bmatrix}$$

1. *A+B*
2. *3A*
3. *2A-3B*

MATRIX MULTIPLICATION

The **product** of an $m \times n$ matrix A and an $n \times k$ matrix B is the $m \times k$ matrix AB, which is computed as follows. To find the element of AB in the ith row and jth column, multiply each element in the ith row of A by the corresponding element in the jth column of B. The sum of these products will give the element of row i, column j in AB.



From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.803

MULTIPLYING MATRICES

If possible, determine the matrix products *AB* and *BA*:

1. $A=\begin{bmatrix}-3 & 5\\ 2 & 7\end{bmatrix}$ $B=\begin{bmatrix}-1 & 2\\ 0 & 7\end{bmatrix}$

2. $A=\begin{bmatrix}2 & 1 & -1\\ 0 & 2 & 1\\ 3 & 2 & -1\end{bmatrix}$ $B=\begin{bmatrix}1 & 0\\ 2 & -1\\ 3 & 1\end{bmatrix}$

3. $A=\begin{bmatrix}3 & -1\\ 2 & -2\\ 0 & 4\end{bmatrix}$ $B=\begin{bmatrix}1 & -4 & 0\\ -1 & 3 & 2\end{bmatrix}$

THE $n \times n$ IDENTITY MATRIX

The $n \times n$ **identity matrix**, denoted I_n, has only 1's on its main diagonal and 0's elsewhere.

INVERSE OF A SQUARE MATRIX

Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix, denoted A^{-1}, that satisfies

$$A^{-1}A = I_n \quad \text{and} \quad AA^{-1} = I_n,$$

then A^{-1} is the **inverse** of A.

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.813

INVERSES

Find the inverse of *A* without a calculator:

1. $A = \begin{bmatrix} -2 & 4 \\ -5 & 9 \end{bmatrix}$

2. $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

EXAMPLE

Write the system in the form $AX = B$ and solve the system by computing $X = A^{-1}B$

1. $\begin{cases} 2x + y = 4 \\ -x + 2y = -1 \end{cases}$

2. $\begin{cases} 17x - 22y - 19z = -25.2 \\ 3x + 13y - 9z = 105.9 \\ x - 2y + 6.1z = -23.55 \end{cases}$

DETERMINANT OF A 2 × 2 MATRIX

The **determinant** of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a real number defined by

$$\det A = ad - cb.$$

INVERTIBLE MATRIX

A square matrix A is invertible if and only if $\det A \neq 0$.

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.825

EXAMPLES

Determine if the inverse exists by computing the determinant of the matrix *A*:

1. $A = \begin{bmatrix} 3 & 1 \\ 7 & -2 \end{bmatrix}$

2. $A = \begin{bmatrix} -4 & 10 \\ 2 & -5 \end{bmatrix}$

MINORS AND COFACTORS

The **minor**, denoted by M_{ij}, for element a_{ij} in the square matrix A is the real number computed by performing the following steps.

STEP 1: Delete the ith row and jth column from the matrix A.

STEP 2: M_{ij} is equal to the determinant of the resulting matrix.

The **cofactor**, denoted A_{ij}, for a_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$.

Example: Find the minor M_{11} and the cofactor A_{11} for the matrix

$$A = \begin{bmatrix} -8 & 0 & 4 \\ 4 & -6 & 7 \\ 2 & -3 & 5 \end{bmatrix}$$

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.826

DETERMINANT OF A MATRIX USING THE METHOD OF COFACTORS

Multiply each element in any row or column of the matrix by its cofactor. The sum of the products is equal to the determinant.

Example: Find det *A* for

$$A = \begin{bmatrix} -8 & 0 & 4 \\ 4 & -6 & 7 \\ 2 & -3 & 5 \end{bmatrix}$$

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.827