OBJECTIVE 4

Exponential & Log Functions

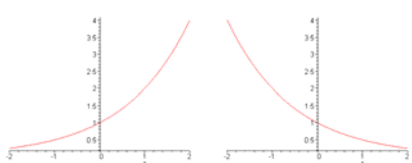
EXPONENTIAL FORM

An exponential function is a function of the form

$$f(x) = b^x$$

where $b > 0$ and $b \neq 1$.

SHAPE OF $y = b^x$



$b > 1$

increasing

$0 < b < 1$

decreasing

PROPERTIES OF THE BASIC EXPONENTIAL GRAPH

Domain is $(-\infty, \infty)$

Range is $(0, \infty)$

Horizontal Asymptote is $y = 0$

y-intercept is $(0, 1)$

DEFINITION OF e

e is an irrational number (like π)

$$e \approx 2.71828182845$$

The natural exponential function is

$$f(x) = e^x$$

GRAPHING EXPONENTIALS

Graph

1. $y = 2^x$
2. $y = 2^{-x}$
3. $y = 2^x+1$
4. $y = 2^{x+1}$
5. $y = 2^{x-3}-2$

EXPONENTIAL APPLICATIONS

Margaret Madison, DDS, estimates that her dental equipment loses one-sixth of its value each year.

a. Determine the value of an x-ray machine after 5 years if it cost \$216 thousand new.

b. Determine how long until the machine is worth less than \$125 thousand.

SOLVING AN EXPONENTIAL EQUATION

Solve for *x*:

1. $81 = 27^x$
2. $\left(\frac{1}{2}\right)^{3x} = 8^{x-2}$
3. $9^{x-1} = 27$
4. $25^{3x} = 125^{x-2}$

LOGARITHMS

The base *b* logarithm of *x*, written $\log_b x$ ($b > 0$ and $b \neq 1$), is the exponent to which *b* must be raised in order to yield *x*.

$$y = \log_b x \text{ is equivalent to } b^y = x$$

LOGARITHMS

$2^3 = 8$ so $\log_2 8 = 3$

$3^2 = 9$ so $\log_3 9 = 2$

$5^3 = 125$ so $\log_5 125 = 3$

LOGARITHMS

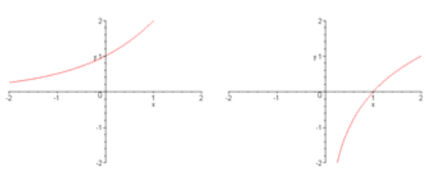
$\log_2 32 = 5$ because $2^5 = 32$

$\log_{10} 0.01 = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$

$\log_9 3 = \frac{1}{2}$ because $9^{1/2} = \sqrt{9} = 3$

$f(x) = \log_a x$ IS THE INVERSE FUNCTION OF $f(x) = a^x$

Obtain the graph of the inverse by reflecting across $y = x$



$f(x) = a^x$ $\qquad$ $f(x) = \log_a x$

GRAPH PROPERTIES

$f(x) = a^x$

- Domain $(-\infty, \infty)$
- Range $(0, \infty)$
- H. A. $y = 0$
- $(0,1)$

$f(x) = \log_a x$

- Domain $(0, \infty)$
- Range $(-\infty, \infty)$
- V. A. $x = 0$
- $(1,0)$

GRAPHING LOGARITHMS

Graph $y = \log_2 x$

Graph $y = \log_2 (x + 5)$

Graph $y = \log_2 (x) + 5$

Graph $y = -3\log_2 (x + 1)$

PROPERTIES

$\log_b b = 1$

$\log_b 1 = 0$

$\log_b b^x = x$

$b^{\log_b x} = x$

LOGARITHMIC PROPERTIES

$\log_2 1 = 0$

$\log_2 2 = 1$

$\log_2 2^3 = 3$

$2^{\log_2 5} = 5$

COMMON LOGARITHM

The common logarithm is base 10. We often write it omitting the base:

$$\log x = \log_{10} x$$

NATURAL LOGARITHM

The natural logarithm is base e. We write it as:

$$\ln x = \log_e x$$

SOLVING EQUATIONS WITH LOGARITHMS

Solve $4^x = 11$

Solve $2(10^x) = 66$

Solve $\log_4 x = 3.7$

Solve $16 - 4\ln 3x = 2$

PROPERTIES OF LOGARITHMS

If x, y, and $b > 0$, then

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b x^k = k \log_b x$$

EXPANDING LOGARITHMS

Expand the expression. If possible, write your answer without exponents.

1. $\ln \frac{xy}{z}$
2. $\log_2 \frac{32}{xy^2}$
3. $\log \sqrt{\frac{xy^2}{z}}$
4. $\log\left(\frac{2x-1}{5xy}\right)$

EXPANDING LOGARITHMS

Expand the expression. If possible, write your answer without exponents.

1. $\ln\sqrt[3]{pq}$
2. $\ln\dfrac{m^2}{n^3}$
3. $\log_3\sqrt{\dfrac{3-v}{2v}}$
4. $\ln\left(\dfrac{7x\sqrt{3-4x}}{2(x-1)^3}\right)$

COMBINING LOGARITHMS

Write the expression as a logarithm of a single expression.

1. $\log_6 45 + 3\log_6 b$
2. $\log_3 x + \dfrac{1}{2}\log_3(x+3) - \dfrac{1}{3}\log_3(x-4)$

COMBINING LOGARITHMS

Write the expression as a logarithm of a single expression.

1. $\log_2 7 + \log_2 6$
2. $\log_3(x^2-25) - \log_3(x+5)$

CHANGE OF BASE FORMULA

Let x, $a \neq 1$, and $b \neq 1$ be positive real numbers. Then

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Example: Evaluate

$$\log_6 0.77$$

STEPS FOR SOLVING EXPONENTIAL EQUATIONS

1. Isolate the power on one side of the equation.
2. Rewrite the equation in logarithmic form.
3. Evaluate the logarithm.
4. Solve for the variable.

SOLVING EXPONENTIAL EQUATIONS

Solve for x

1. $2e^{-x} = 8$
2. $100 - 5(10)^x = 7$
3. $e^{2x} = e^{5x-3}$
4. $30 - 3(0.75)^{x-1} = 29$

SOLVING EXPONENTIAL EQUATIONS

Solve for x

1. $7^{x+2} = 231$
2. $7^{x} = 4^{2x-1}$
3. $\dfrac{80}{1+15e^{-0.06x}} = 50$

STEPS FOR SOLVING LOGARITHMIC EQUATIONS

1. Combine all logarithms using the properties of logarithms.
2. Isolate the logarithm on one side of the equation.
3. Rewrite the equation in exponential form.
4. Simplify and solve for the variable.
5. Check your answer!!! You may have answers that don't work.

SOLVING LOGARITHMIC EQUATIONS

Solve for x

1. $5\ln x = 10$
2. $\log_3(1-x) = 1$
3. $\log x^5 = 4 + 3\log x$
4. $\ln(x^2-4) - \ln(x+2) = \ln(3-x)$

SOLVING LOGARITHMIC EQUATIONS

Solve for *x*

1. $\frac{3}{4}\ln(4x) - 6.9 = -5.1$
2. $\log(3x - 13) = 2 - \log x$
3. $\log(x + 14) - \log x = \log(x + 6)$

COMPOUND INTEREST

The amount accumulated in an account bearing interest compounded n times annually is

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where P = principal invested

r = interest rate (as a decimal)

t = time in years

COMPOUND INTEREST

Suppose \$5000 is invested at an interest rate of 8%. Find the amount in the account after ten years if the interest is compounded

a. annually

b. semiannually

c. daily

COMPOUND INTEREST

What principal should be deposited at 8.375% compounded monthly to ensure the account will be worth $20,000 in 10 years?

CONTINUOUSLY COMPOUNDED INTEREST

The amount accumulated in an account bearing interest compounded continuously is

$$A(t) = Pe^{rt}$$

where P = principal invested

r = interest rate (as a decimal)

t = time in years

CONTINUOUSLY COMPOUNDED INTEREST

$5000 is invested at an interest rate of 8%, compounded continuously. How much is in the account after 10 years?

CONTINUOUSLY COMPOUNDED INTEREST

What principal should be deposited at 8.375% compounded continuously to ensure the account will be worth \$20,000 in 10 years?

EXPONENTIAL GROWTH AND DECAY

Exponential growth of a population is given by the formula

$$P(t) = P_0 e^{kt}$$

where

P_0 = initial size of the population

k = relative rate of growth (positive) or decay (negative)

t = time

EXPONENTIAL GROWTH

The population of Phoenix, Arizona, in 2000 was 1.3 million and growing continuously at a 3% rate.

a. Assuming this trend continues, estimate the population of Phoenix in 2010.

b. Determine graphically or numerically when this population might reach 2 million.

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.416, problem 98.

EXPONENTIAL DECAY

The radioactive element americium-241 has a half-life of 432 years and although extremely small amounts are used (about 0.0002 g), it is the most vital component of standard household smoke detectors. How many years will it take a 10-g mass of americium-241 to decay to 2.7 g?