OBJECTIVE 3

Polynomial & Rational Functions

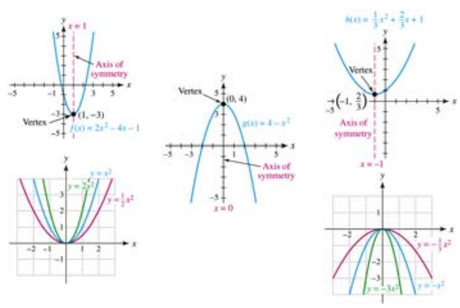
FEATURES OF THE GRAPH OF A QUADRATIC

The graph of $y = ax^2 + bx + c$ is a parabola

The vertex is the highest or lowest point on the graph

The axis of symmetry passes through the vertex

EXAMPLES OF DIFFERENT PARABOLAS



From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.173.

GRAPH OF $y = ax^2 + bx + c$

The coefficient of x^2, a, determines the width of the graph.

- If $|a| < 1$, the graph is more wide than the graph of $y = x^2$
- If $|a| > 1$, the graph is more narrow than the graph of $y = x^2$

- If $a > 0$, the parabola opens up
- If $a < 0$, the parabola opens down

VERTEX FORM

The vertex form of a quadratic equation is

$$y = a(x - h)^2 + k$$

where the vertex of the graph is (h, k)

SKETCHING A QUADRATIC

Sketch the graph of

$$f(x) = (x + 1)^2 - 2$$

STEPS FOR COMPLETING THE SQUARE

$$y = ax^2 + bx + c$$

1. Make sure the coefficient of x^2 is $+1$. If it is not, factor a out of the x terms only.
2. Square half the coefficient of x and add to the x terms and subtract this quantity from c (remember to multiply by a, if necessary).
3. Write the result as a perfect square and simplify the constant terms.

SKETCHING QUADRATICS

Sketch the graph of the function. Identify the x-intercepts, vertex, and maximum or minimum of the function.

1. $f(x) = x^2 + 10x + 7$
2. $f(x) = 3x^2 + 6x + 2$
3. $f(x) = -x^2 + 2x + 1$
4. $f(x) = -\frac{1}{2}x^2 + x + 1$

OPTIMIZATION

A kitchen appliance manufacturer can produce up to 200 appliances per day. The profit made from the sale of these machines can be modeled by the function $P(x) = -0.5x^2 + 175x - 3300$, where $P(x)$ is the profit in dollars, and x is the number of appliances made and sold. Based on this model,

a. Find the y-intercept and explain what it means in this context.
b. Find the x-intercepts and explain what they mean in this context.
c. Determine the domain of the function and explain its significance.
d. How many should be sold to maximize profit? What is the maximum profit?

DEGREE OF A POLYNOMIAL

In a polynomial containing only one variable, the greatest exponent that appears on the variable is called the degree of the polynomial

Examples: $x^8 - 3x^2 + 1$ is degree 8

$-2x^4 - x^6$ is degree 6

5 is degee 0

x is degree 1

DIVISION BY A MONOMIAL

Divide each term of the polynomial by the monomial

Example: Divide $\dfrac{5x^3 - 10x^2 + 5x}{15x^2}$

LONG DIVISION OF POLYNOMIALS

The process of long division of polynomials is similar to long division of numbers.

Divide

1. $(12x^3 - 14x^2 + 7x - 7) \div (3x - 2)$
2. $(x^3 - x^2 + 2x - 3) \div (x^2 + 3)$
3. $(3x^4 - 2x^2 - 5) \div (3x^2 - 5)$

SYNTHETIC DIVISION

Synthetic division is a short cut to the process of long division.

Divide

1. $(x^3 - 2x^2 - x + 3) \div (x + 1)$
2. $(x^4 + 3x^3 - 4x + 1) \div (x + 2)$

REMAINDER THEOREM

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If a polynomial $f(x)$ is divided by $x - k$, the remainder is $f(k)$.

Example: Use the remainder theorem to find the remainder when $f(x) = -4x^2 + 6x - 7$ is divided by $x + 4$

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.279

FACTOR THEOREM

FACTOR THEOREM
A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

Example: Use the factor theorem to decide if $x - ½$ is a factor of $f(x) = 2x^4 - 11x^3 + 9x^2 + 14x$

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.279

FACTORED FORM

For the polynomial below, -2 is a zero. Express $f(x)$ as a product of linear factors.

$$f(x) = 2x^3 + x^2 - 11x - 10$$

RATIONAL ZEROS

RATIONAL ZERO TEST

Let $f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$, where $a_n \neq 0$, represent a polynomial function f with integer coefficients. If $\frac{p}{q}$ is a rational number written in lowest terms and if $\frac{p}{q}$ is a zero of f, then p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

Find all rational zeros of $f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$.

GRAPHS OF POLYNOMIALS

Degree, *x*-intercepts, and turning points

The graph of a polynomial function of degree $n \geq 1$ has at most n x-intercepts and at most $n - 1$ turning points.

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.262

END BEHAVIOR

A polynomial of odd degree with a positive leading coefficient has negative *y*-values for large negative *x*-values, and positive *y*-values for large positive *x*-values.

A polynomial of even degree with a positive leading coefficient has positive *y*-values for both large positive and large negative *x*-values.

NUMBER OF ZEROS OF POLYNOMIALS

Fundamental Theorem of Algebra

The polynomial $f(x)$ of degree $n \geq 1$ has at least one complex zero.

Number of Zeros Theorem

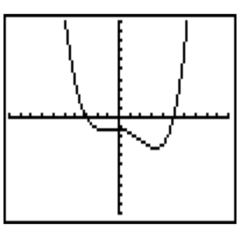
A polynomial of degree n has at most n distinct zeros.

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.296-7

EXAMPLE

The graph provided is a degree 4 polynomial.

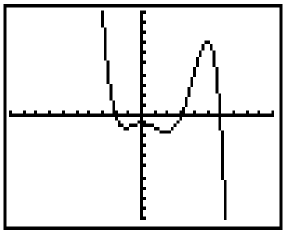
1. Identify whether the leading coefficient is positive or negative.
2. How many real and how many imaginary zeros does the polynomial have?



EXAMPLE

The graph provided is a degree 5 polynomial.

1. Identify whether the leading coefficient is positive or negative.
2. How many real and how many imaginary zeros does the polynomial have?



COMPLEX ZEROS OF POLYNOMIALS

Conjugate Zeros Theorem

If a polynomial $f(x)$ has only real coefficients and if $a + bi$ is a zero of $f(x)$, then the conjugate $a - bi$ is also a zero of $f(x)$.

From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.299

EXAMPLES

1. Find the equation of a a degree 3 polynomial with leading coefficient – ¾ and zeros – 3i and 2/5.

2. Given that 2i is one zero, find all the zeros of $f(x) = x^4 + 2x^3 + 8x^2 + 8x + 16$

3. Find all the zeros of $f(x) = x^3 + 2x^2 + 16x + 32$

MULTIPLICITY OF ZEROS

If a polynomial has a zero of odd multiplicity, the graph crosses the *x*-axis at that point. If a polynomial has a zero of even multiplicity, the graph "bounces off" the *x*-axis at that point.

MULTIPLICITY

For each polynomial,

a. Find the x- and y-intercepts.

b. Determine the multiplicity of each zero.

c. Sketch a graph by hand.

1. $f(x) = -3(x-1)^3$

2. $f(x) = x^2(x+2)(x-2)$

RATIONAL FUNCTIONS

A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

Examples of rational functions: $f(x) = \frac{1}{x}$

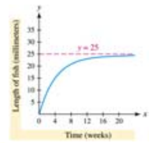
$$f(x) = \frac{x^2-3}{2x+1}$$

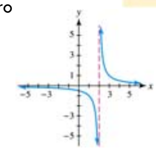
$$g(x) = \frac{5x-2}{x^2-3x-4}$$

ASYMPTOTES

A rational function may have both vertical and horizontal asymptotes.

A vertical asymptote occurs anywhere the denominator is zero and the numerator is not.





From *Precalculus with Modeling and Visualization* 3rd ed. by Rockswold, 2006, p.207-8

HORIZONTAL ASYMPTOTES

If the degree of the denominator is larger than the degree of the numerator, the horizontal asymptote is $y = 0$.

If the degree of the numerator is larger than the degree of the denominator, there is no horizontal asymptote.

If the degrees are the same, the horizontal asymptote is given by the ratio of the leading coefficients. That is, if

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0}$$

$$\text{then} \quad y = \frac{a_n}{b_n}$$

SLANT ASYMPTOTE

A slant asymptote (or oblique asymptote) may occur when the degree of the numerator is exactly one more than the degree of the denominator.

The equation of the slant asymptote is the quotient of performing long division.

TO GRAPH A RATIONAL FUNCTION

1. Find the vertical asymptotes by setting the denominator equal to zero.
2. Find the horizontal asymptote by comparing the degrees of the numerator & denominator.
3. Find the y-intercept by setting $x = 0$.
4. Find the x-intercept by setting the numerator equal to zero.
5. Sketch the graph (plot points to help, if necessary).

GRAPHING RATIONAL FUNCTIONS

Sketch the function:

1. $y = \frac{2x+3}{x+1}$
2. $y = \frac{x^2-4}{x^2-x-6}$
3. $y = \frac{4x^2+4x+1}{2x+1}$

GRAPHING RATIONAL FUNCTIONS

Sketch the function:

1. $y = \frac{2x-4}{x^2+2x-3}$
2. $y = \frac{x^2-2x+1}{x+5}$

PROCEDURE TO SOLVE INEQUALITIES ALGEBRAICALLY

1. Get a 0 on one side of the inequality
2. Factor completely (numerator & denominator)
3. Plot the roots of all factors on a number line
4. Pick a value from each interval on the number line and plug it into the expression to find the sign
5. Choose the intervals carrying the appropriate sign
6. Write your answer in interval notation

REMEMBER!

In a rational function, the zeros of the denominator CANNOT be included in the solution set ever.

SOLVING POLYNOMIAL INEQUALITIES

Solve the inequality and present your answer in interval notation:

1. $8x^3 < 27$
2. $2x^3 \leq 3x^2 + 5x$
3. $4x^4 - 5x^2 - 9 \geq 0$

SOLVING RATIONAL INEQUALITIES

Solve the inequality and present your answer in interval notation:

1. $\frac{x}{x^2-1} \geq 0$
2. $\frac{x+1}{4-2x} \geq 1$
3. $\frac{3}{2-x} > \frac{x}{2+x}$

SOLVING INEQUALITIES

Solve the inequality and present your answer in interval notation:

1. $-x^2 + 3x < 3$
2. $x^3 + x^2 - 5x + 3 \leq 0$
3. $\frac{x-3}{x-6} \leq \frac{x+1}{x+4}$

POLYNOMIAL INEQUALITIES

Given the graph of $f(x)$, solve the inequality $f(x) \leq 0$.