Asymptotes

We deal with two types of asymptotes: vertical asymptotes and horizontal asymptotes.

Vertical Asymptotes

There are two functions we will encounter that may have vertical asymptotes: rational functions and logarithmic functions.

Rational Functions

As with finding the domain, we start by setting the denominator equal to zero. The values of x that we obtain by doing so are the possibilities of the locations of vertical asymptotes. But we have one extra step–we must plug these values into the numerator to make sure that we don't have a zero there at the same time.

Example 1 *Find the vertical asymptotes of the function* $f(x) = \frac{2-x}{x^2-4x-21}$.

Solution 1 *We start with the zeros of the denominator.*

$$x^2 - 4x - 21 = 0$$

$$(x-7)(x+3) = 0$$

$$x - 7 = 0 \text{ and } x + 3 = 0$$

$$x = 7 \text{ and } x = -3$$

Now plug each into the numerator.

$$2 - 7 \neq 0 \text{ and } 2 - (-3) \neq 0$$

Each works, so here we have two asymptotes: $x = 7$ *and* $x = -3$*. Note: our final answer included the* $x =$ *before the number because we are giving the equation of vertical lines. Your answers should always be given this way.*

Example 2 *Find the vertical asymptotes of the function* $f(x) = \frac{7-x}{x^2-4x-21}$.

Solution 2 *Again, we start with the zeros of the denominator.*

$$x^2 - 4x - 21 = 0$$

$$(x-7)(x+3) = 0$$

$$x - 7 = 0 \text{ and } x + 3 = 0$$

$$x = 7 \text{ and } x = -3$$

Now plug each into the numerator.

$$7 - 7 = 0 \text{ and } 2 - (-3) \neq 0$$

Only the second is nonzero, so here we have one asymptote: $x = -3$. *The other gives us a hole in the graph.*

Logarithms

With logarithms, the vertical asymptotes occur where the argument of the logarithm is zero. This gives us the procedure to find the asymptote: set the inside of the logarithm to zero and solve for x.

Example 3 *Find the vertical asymptote for* $f(x) = \log(2-x)$.

Solution 3 *Set the inside of the logarithm to zero and solve for* x.

$$2 - x = 0$$

$$2 = x$$

Thus, the equation of our vertical asymptote is $x = 2$.

Horizontal asymptotes

Horizontal asymptotes are used to describe the end behavior of some graphs. These are lines that the function gets close to as it moves out on the ends of the graph (big positive values of x and big negative values of x). There are two functions we will encounter that may have horizontal asymptotes: rational functions and exponential functions.

Rational functions

You learn how to find the horizontal asymptote of a rational function when you learn about limits in calculus. Once we have discussed that section in class, **you will no longer use the following shortcut method**. You may, however, use it to check yourself so it is worth knowing.

First we look at the degree (highest exponent) in the numerator and denominator of the fraction. Then we use the following rules:

- If the degree of the numerator is bigger, there is no horizontal asymptote.
- If the degree of the denominator is bigger, the horizontal asymptote is always the line $y = 0$ (the x axis).
- If the degrees are the same in the top and bottom, the horizontal asymptote is the ratio of the leading coefficients (the coefficients in front of the highest powers).

Example 4 *Find the horizontal asymptote of* $f(x) = \frac{3-2x-x^8}{x^2+x+1}$.

Solution 4 *The degree of the numerator is* 8. *The degree of the denominator is* 2. *The numerator is bigger. There is no horizontal asymptote.*

Example 5 *Find the horizontal asymptote of* $f(x) = \frac{3-2x}{x^2+x+1}$.

Solution 5 *The degree of the numerator is* 1. *The degree of the denominator is* 2. *The denominator is bigger. The horizontal asymptote is* $y = 0$.

Example 6 *Find the horizontal asymptote of* $f(x) = \frac{3-2x-x^2}{x^2+x+1}$.

Solution 6 *The degree of the numerator is* 2. *The degree of the denominator is* 2. *They are the same. The horizontal asymptote is the coefficient of* x^2 *in the numerator divided by the coefficient of* x^2 *in the denominator. That is, the horizontal asymptote is the line* $y = \frac{-1}{1}$ *or* $y = -1$.

Example 7 *Find the horizontal asymptote of* $f(x) = \frac{3-2x}{5x+1}$.

Solution 7 *The degree of the numerator is* 1. *The degree of the denominator is* 12. *They are the same. The horizontal asymptote is* $y = \frac{-2}{5}$.

Exponential functions

The line $y = 0$ is a horizontal asymptote for exponential functions of the form $y = a^x$. Know this! We can change this asymptote by adding or subtracting real numbers to this basic function (recall, this shifts the graph up or down).

Example 8 *Find the horizontal asymptote of* $y = e^x$.

Solution 8 *This is the basic form given above. The horizontal asymptote is* $y = 0$.

Example 9 *Find the horizontal asymptote of* $y = e^x + 5$.

Solution 9 *This is the basic form given above shifted up five units. The horizontal asymptote is* $y = 5$.

Example 10 *Find the horizontal asymptote of* $y = e^x - 3$.

Solution 10 *This is the basic form given above shifted down three units. The horizontal asymptote is* $y = -3$.