1. Determine whether the functions are linearly independent: x^2, x^2+1, x^2-1.

2. If $y_1 = e^{5x}$ is a solution of $y'' - 25y = 0$, use the reduction of order technique (formula) to find a second solution $y_2(x)$.

3. Solve $y'' - 3y' + 2y = 0$.

4. Solve $y'' - 10y' + 25y = 0$.

5. Solve the IVP $\frac{d^2y}{d\theta^2} + y = 0$, $y\left(\frac{\pi}{3}\right) = 0$, $y'\left(\frac{\pi}{3}\right) = 2$.

6. Solve $y'' + 2y' = 2x + 5 - e^{-2x}$ by undetermined coefficients.

7. Solve $y'' + y = \sec^2 x$ by variation of parameters.

8. Solve $x^2y'' - 7xy' + 41y = 0$.

1. Determine whether the functions are linearly independent: x^2, x^2+1, x^2-1.

$$W = \begin{vmatrix} x^2 & x^2+1 & x^2-1 \\ 2x & 2x & 2x \\ 2 & 2 & 2 \end{vmatrix}$$

Expanding along the first row,

$$W = x^2(2x \cdot 2 - 2x \cdot 2) - (x^2+1)(2x \cdot 2 - 2x \cdot 2) + (x^2-1)(2x \cdot 2 - 2x \cdot 2) = 0$$

Thus, the functions are linearly dependent.

2. If $y_1 = e^{5x}$ is a solution of $y'' - 25y = 0$, use the reduction of order technique (formula) to find a second solution $y_2(x)$.

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$y_2 = e^{5x} \int \frac{e^{-\int 0\,dx}}{e^{10x}}\,dx$$

$$y_2 = e^{5x} \int \frac{e^0}{e^{10x}}\,dx$$

$$y_2 = e^{5x} \int \frac{1}{e^{10x}}\,dx$$

$$y_2 = e^{5x} \int e^{-10x}\,dx$$

$$y_2 = e^{5x} \left(\frac{-1}{10} e^{-10x}\right)$$

$$y_2 = \frac{-1}{10} e^{-5x}$$

3. Solve $y'' - 3y' + 2y = 0$.

The auxiliary equation is $m^2 - 3m + 2 = 0$ so $m = 1,\ 2$ and $y = C_1 e^x + C_2 e^{2x}$.

4. Solve $y'' - 10y' + 25y = 0$.

The auxiliary equation is $m^2 - 10m + 25 = 0$ so $m = 5,\ 5$ and $y = C_1e^{5x} + C_2xe^{5x}$.

5. Solve the IVP $\frac{d^2y}{d\theta^2} + y = 0$, $y\left(\frac{\pi}{3}\right) = 0$, $y'\left(\frac{\pi}{3}\right) = 2$.

The auxiliary equation is $m^2+1 = 0$ so $m = \pm i$ and $y = C_1 \cos\theta + C_2 \sin\theta$. Then, $y' = -C_1 \sin\theta + C_2 \cos\theta$. Applying the initial conditions, we find

$$0 = C_1 \cos\frac{\pi}{3} + C_2 \sin\frac{\pi}{3}$$

$$2 = -C_1 \sin\frac{\pi}{3} + C_2 \cos\frac{\pi}{3}$$

so the system of equations is

$$0 = C_1\left(\frac{1}{2}\right) + C_2\left(\frac{\sqrt{3}}{2}\right)$$

$$2 = -C_1\left(\frac{\sqrt{3}}{2}\right) + C_2\left(\frac{1}{2}\right)$$

or

$$0 = C_1 + \sqrt{3}C_2$$
$$2 = -\sqrt{3}C_1 + C_2$$

solving this system we find $C_1 = -\frac{\sqrt{3}}{2}$ and $C_2 = \frac{1}{2}$. So the solution is

$$y = -\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta$$

6. Solve $y'' + 2y' = 2x + 5 - e^{-2x}$ by undetermined coefficients.

Solving $y''+2y' = 0$ uses the auxiliary equation $m^2+2m = 0$ so $m = 0,\ -2$ and $y_c = C_1 + C_2e^{-2x}$. Assume $y_p = ax^2 + bx + cxe^{-2x}$ (both the polynomial and the exponential were multiplied by x since each had a term that appeared in y_c). Then $y_p' = 2ax + b + ce^{-2x} - 2cxe^{-2x}$ and $y'' = 2a - 4ce^{-2x} + 4cxe^{-2x}$. Substituting into the DE,

$$2x + 5 - e^{-2x} = 2a - 4ce^{-2x} + 4cxe^{-2x} + 2(2ax + b + ce^{-2x} - 2cxe^{-2x})$$

$$2x + 5 - e^{-2x} = 2a + 2b + 4ax - 2ce^{-2x}$$

from which we get the system of equations $2a+2b=5$, $4a=2$, and $-2c=-1$. Solving this system gives $a=\frac{1}{2}$, $b=2$, and $c=\frac{1}{2}$ so $y_p=\frac{1}{2}x^2+2x+\frac{1}{2}xe^{-2x}$. Thus, the solution is $y=C_1+C_2e^{-2x}+\frac{1}{2}x^2+2x+\frac{1}{2}xe^{-2x}$

7. Solve $y''+y=\sec^2 x$ by variation of parameters.

Solving $y''+y=0$ has the auxiliary equation $m^2+1=0$ so $m=\pm i$ and $y_c=C_1\cos x+C_2\sin x$. Now, $y_1=\cos x$, $y_2=\sin x$, and $f(x)=\sec^2 x$. So, assuming $y_p=u_1y_1+u_2y_2$,

$$W=\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}=\cos^2 x+\sin^2 x=1$$

$$W_1=\begin{vmatrix} 0 & \sin x \\ \sec^2 x & \cos x \end{vmatrix}=0+\sin x\sec^2 x=\sec x\tan x$$

$$W_2=\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec^2 x \end{vmatrix}=\cos x\sec^2 x-0=\sec x$$

Now we have $u_1'=\frac{W_1}{W}=\sec x\tan x$ and $u_2'=\frac{W_2}{W}=\sec x$. Then integrating gives

$$u_1=\int \sec x\tan x=\sec x$$

$$u_2=\int \sec x=\ln|\sec x+\tan x|$$

So it follows that $y_p=\cos x\sec x+\sin x\ln|\sec x+\tan x|=1+\sin x+\ln|\sec x+\tan x|$. Then the solution is

$$y=C_1\cos x+C_2\sin x+1+\sin x\ln|\sec x+\tan x|$$

8. Solve $x^2y''-7xy'+41y=0$.

Letting $x=e^t$ we have $dx=e^tdt$ or $\frac{dt}{dx}=e^{-t}$ then

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=e^{-t}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)\frac{dt}{dx}=\left(e^{-t}\frac{d^2y}{dt^2}-e^{-t}\frac{dy}{dt}\right)e^{-t}=e^{-2t}\frac{d^2y}{dt^2}-e^{-2t}\frac{dy}{dt}$$

and substituting into the DE gives

$$
\begin{aligned}
0 = x^2y'' - 7xy' + 41y &= \left(e^t\right)^2 \left(e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}\right) - 7e^t \left(e^{-t}\frac{dy}{dt}\right) + 41y \\
&= e^{2t} \left(e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}\right) - 7\frac{dy}{dt} + 41y \\
&= \frac{d^2y}{dt^2} - \frac{dy}{dt} - 7\frac{dy}{dt} + 41y \\
&= \frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 41y
\end{aligned}
$$

so solving $\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 41y = 0$ we have the auxiliary equation $m^2 - 8m + 41 = 0$ which gives $m = 4 \pm 5i$. So, $y = C_1e^{4t}\cos 5t + C_2e^{4t}\sin 5t$ and substituting $t = \ln x$ we have $y = C_1e^{4\ln x}\cos(5\ln x) + C_2e^{4\ln x}\sin(5\ln x)$ or

$$y = C_1x^4\cos(5\ln x) + C_2x^4\sin(5\ln x)$$