

MAP 2302 Sample Problems, Exam 3

1. Use the definition of the Laplace transform to find $\mathcal{L}\{f(t)\}$ for

$$f(t) = \begin{cases} t, & 0 \leq t < 4 \\ 0, & t \geq 4 \end{cases}$$

2. Use partial fractions to find inverse Laplace transforms of the following:

(a) $\mathcal{L}^{-1}\left\{\frac{s-11}{(s+1)(s-2)(s-3)}\right\}$

(b) $\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)(s^2+4)}\right\}$

3. Use the translation theorems to find:

(a) $\mathcal{L}\left\{\frac{1}{2}t^2e^{-2t}\right\}$

(b) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}$

(c) $\mathcal{L}\{e^{-3t}U(t-\pi)\}$

(d) $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^{3/2}}\right\}$

4. Write $f(t) = \begin{cases} 2t+3, & 0 \leq t < 9 \\ -2, & t \geq 9 \end{cases}$ in terms of unit step functions. Find the Laplace transform of the function.

5. Find $\mathcal{L}\{t^2 \cosh 3t\}$.

6. Use Laplace transforms to solve the initial-value problem

$$y'' + 9y = 20e^{-t}, \quad y(0) = 0, \quad y'(0) = 1.$$

7. Find the Laplace transform of the function whose graph is Figure 7.42 on page 348 of the textbook (that is problem number 26 in 7.4).

8. Find $\mathcal{L}\left\{\int_0^t \tau^2 \cos(t-\tau) dt\right\}$.

9. Solve the initial-value problem

$$y'' + 4y' + 4y = 6\delta(t-2), \quad y(0) = 0, \quad y'(0) = 0.$$

1. Use the definition of the Laplace transform to find $\mathcal{L}\{f(t)\}$ for

$$f(t) = \begin{cases} t, & 0 \leq t < 4 \\ 0, & t \geq 4 \end{cases}$$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^\infty f(t)e^{-st} dt \\ &= \int_0^4 te^{-st} dt \\ &= -\frac{t}{s}e^{-st} \Big|_0^4 + \frac{1}{s} \int_0^4 e^{-st} dt \\ &= -\frac{4}{s}e^{-4s} - \frac{1}{s^2}e^{-st} \Big|_0^4 \\ &= -\frac{4}{s}e^{-4s} - \frac{1}{s^2}e^{-4s} + \frac{1}{s^2} \\ &= \frac{1}{s^2} - \frac{4s+1}{s^2}e^{-4s} \end{aligned}$$

2. Use partial fractions to find inverse Laplace transforms of the following:

$$(a) \quad \mathcal{L}^{-1}\left\{\frac{s-11}{(s+1)(s-2)(s-3)}\right\}$$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s-11}{(s+1)(s-2)(s-3)}\right\} &= \mathcal{L}^{-1}\left\{\frac{-1}{s+1} + \frac{3}{s-2} - \frac{2}{s-3}\right\} \\ &= -e^{-t} + 3e^{2t} - 2e^{3t} \end{aligned}$$

$$(b) \quad \mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)(s^2+4)}\right\}$$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+1)(s^2+4)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{s+1}{s^2+1} - \frac{1}{3}\frac{s+1}{s^2+4}\right\} \\ &= \frac{1}{3}(\cos t + \sin t) - \frac{1}{3}\cos 2t - \frac{1}{6}\sin 2t \end{aligned}$$

3. Use the translation theorems to find:

$$(a) \quad \mathcal{L}\left\{\frac{1}{2}t^2e^{-2t}\right\}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2e^{-2t}\right\} = \frac{1}{2}\mathcal{L}\{t^2\}|_{s \rightarrow s+2} = \frac{1}{(s+2)^3}$$

$$(b) \quad \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+1/2}{(s+1/2)^2+3/4} + \frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+3/4}\right\} \\ &= e^{-t/2} \left(\cos \sqrt{3}t/2 + \frac{1}{\sqrt{3}} \sin \sqrt{3}t/2 \right)\end{aligned}$$

$$(c) \quad \mathcal{L}\{e^{-3t}U(t-\pi)\}$$

$$\mathcal{L}\{e^{-3t}U(t-\pi)\} = \mathcal{L}\{U(t-\pi)\}|_{s \rightarrow s+3} = \frac{e^{-\pi(s+3)}}{s+3}$$

$$(d) \quad \mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^{3/2}}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^{3/2}}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\mathcal{L}\left\{\frac{2}{\sqrt{\pi}}e^{-t}t^{1/2}\right\}\right\} = \frac{2}{\sqrt{\pi}}e^{-(t-1)}(t-1)^{1/2}U(t-1)$$

4. Write $f(t) = \begin{cases} 2t+3, & 0 \leq t < 9 \\ -2, & t \geq 9 \end{cases}$ in terms of unit step functions. Find the Laplace transform of the function.

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{2t+3 - (2t+5)U(t-9)\} \\ &= \frac{2}{s^2} + \frac{3}{s} - e^{-9s}\mathcal{L}\{2t+23\} \\ &= \frac{2}{s^2} + \frac{3}{s} - e^{-9s}\left(\frac{2}{s^2} + \frac{23}{s}\right)\end{aligned}$$

5. Find $\mathcal{L}\{t^2 \cosh 3t\}$.

$$\mathcal{L}\{t^2 \cosh 3t\} = (-1)^2 \frac{d^2}{ds^2} \left(\frac{s}{s^2-9} \right) = \frac{2s^3 + 54s}{(s^2-9)^3}$$

6. Use Laplace transforms to solve the initial-value problem

$$y'' + 9y = 20e^{-t}, \quad y(0) = 0, \quad y'(0) = 1.$$

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 9Y(s) &= \frac{20}{s+1} \\ (s^2 + 9)Y(s) &= \frac{20}{s+1} + 1 \\ Y(s) &= \frac{20}{(s^2 + 9)(s + 1)} + \frac{1}{s^2 + 9} \\ Y(s) &= \frac{2}{s+1} - \frac{2s}{s^2 + 9} + \frac{3}{s^2 + 9} \\ y(t) &= 2e^{-t} - 2\cos 3t + \sin 3t \end{aligned}$$

7. Find the Laplace transform of the function whose graph is Figure 7.42 on page 314 of the textbook (that is problem number 54 in 7.4).

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t) dt \\ &= \frac{1}{1 - e^{-2\pi s}} \int_0^\pi e^{-st} \sin t dt \\ &= \frac{1}{1 - e^{-2\pi s}} \frac{1 + e^{-\pi s}}{s^2 + 1} \\ &= \frac{1}{(1 - e^{-\pi s})(s^2 + 1)} \end{aligned}$$

8. Find $\mathcal{L}\{\int_0^t \tau^2 \cos(t - \tau) dt\}$.

$$\mathcal{L}\left\{\int_0^t \tau^2 \cos(t - \tau) dt\right\} = \mathcal{L}\{t^2 * \cos t\} = \frac{2}{s^2(s^2 + 1)}$$

9. Solve the initial-value problem

$$y'' + 4y' + 4y = 6\delta(t - 2), \quad y(0) = 0, \quad y'(0) = 0.$$

$$s^2Y(s)-sy(0)-y'(0)+4sY(s)-4y(0)+4Y(s)=6e^{-2s}$$

$$(s^2+4s+4)Y(s)=6e^{-2s}$$

$$Y(s)=\frac{6e^{-2s}}{(s+2)^2}$$

$$y(t)=6\mathcal{L}^{-1}\{e^{-2s}\mathcal{L}\{te^{-2t}\}\}$$

$$y(t)=6(t-2)e^{-2(t-2)}U(t-2)$$